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LXX. *Some Relations between Transient Phenomena in Systems with Similar Frequency Characteristics. By* H. G. BAERWALD. (Communication from the Staff of the Research Laboratories of the General Electrical Company, Limited, Wembley, England *.)

Summary.

THE accurate determination of the transient response to a given excitation in a linear system requires the knowledge of the whole frequency characteristic. In physical systems this knowledge is of limited accuracy, and is restricted to a part of the frequency scale. Some relations between the limitations of a frequency characteristic and the associated transient errors are discussed. The results apply equally to the method of substituting a similar frequency characteristic in order to facilitate the calculations of transients, which is used in technical applications. Upper limits are obtained for the errors of transient responses to prescribed excitations when upper limits are given for the errors of the characteristic in limited frequency ranges, and when some relatively inferior information is supplied about the rest. In most cases the resulting error limits do not apply to the transients directly, but to related quantities obtained by a

* Communicated by C. C. Paterson, M.I.E.E.

mean value process. By the application of this process the frequency ranges of main information are given that additional prominence which is required for obtaining sufficiently definite results.

THE calculation of transients in linear systems is a very familiar subject [1], if the steady state response is assumed to be known for all frequencies. This, however, is never strictly true, since a physical network matches a theoretical design only approximately. The degree of accuracy usually varies considerably with frequency, being, *e. g.*, high for a certain frequency interval and gradually decreasing beyond it, until for remote frequencies data become very vague. The question arises what kind of, and how accurate, information about transient phenomena may be obtained if the knowledge of the frequency characteristic is limited. A similar problem arises if the calculation of transients from a completely prescribed characteristic involves mathematical work which is not in proportion with the physical simplicity or the required accuracy. In such cases an auxiliary function is often found which matches the given characteristic fairly well in the essential frequency ranges, and, when substituted, simplifies the mathematical work. Related measures have been successfully applied to circuit problems [2], but little attention has been paid to the relation between steady state and transient errors. In the following some simple results are given concerning this problem ; they were obtained in connection with calculations of transients in power rectifier circuits, which will be communicated elsewhere.

In the fundamental relations of the symbolic calculus (see [1])

$$\frac{f(z)}{z} = \int_0^\infty e^{-zt} h(t)\, dt \quad \text{(Laplace-Carson integral)}, \quad . \quad . \quad \text{(I.)}$$

[1] Selected references :—J. R. Carson, 'Electric Circuit Theory and Operational Calculus' New York (1926). H. Jeffreys, 'Operational Methods in Mathematics and Physics' (Cambridge, 1927). G. A. Campbell and R. M. Foster, Bell Syst. Tech. Monograph, B. 584, Sept. 1931. B. v. d. Pol and collaborators, Phil. Mag. vii. p. 1153 (1929) ; viii. p. 861 (1929) ; xi. p. 368 (1931) ; xiii. p. 537 (1932) ; *Tijdschr. v. h. Nederl. Radiogenootsch.* vii. p. 18 (March 1935) ; also U.S.R.I. Document A. G. 1934, no. 12, Comm. v. P. Humbert, 'Le Calcul Symbolique' (Paris 1934).

[2] K. Küpfmüller, *Elektr. Nachr. Techn.* v. p. 30 (1928). J. R. Carson, Bell Syst. Tech. Journ. iii. p. 163 (1924) ; iv. p. 265 (1925).

$$h(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tz}\frac{f(z)}{z}\,dz. \quad \text{(Wagner-Bromwich integral),} \qquad \ldots \quad \text{(II.)}$$

summarized by the notation

$$f(z) \doteqdot h(t) \qquad \ldots \ldots \quad \text{(III.)}$$

The "original" $h(t)$ and the "image" function $f(z)$ are interpreted in the conventional manner by considering a response $S_2(t)$ to a single excitation $S_1(t)$ in a linear system. S_1 and S_2 denote any two variables (voltage, current, displacement, etc.), and, according to the superposition principle, any movement is a linear compound of single responses. If $S_1(t)$ varies exponentially,

$$S_1=A_1e^{zt};$$

S_2 does the same, in consequence of linearity;

$$S_2=A_2e^{zt},\ A_2=f(z)A_1.$$

This is true for complex values of z as well, and leads to the interpretation of the analytic function $f(z)$ as ratio of complex amplitudes. On the other hand, $h(t)$ is defined as response to the unit jump

$$D_0(t\,;\,0)=\left\{\begin{matrix}0,\ t<0.\\ 1,\ t>0.\end{matrix}\right\} \qquad \ldots \ldots \quad (1)$$

It follows from definition that $f(z)$ is a real function (*i. e.*, real for real values),

$$\left.\begin{matrix}f(\bar{z})=\bar{f}(z), \text{ especially } u(-iy)=u(iy), v(-iy)=-v(iy)\\ [f=u+iv,\ z=x+iy\,;\ \bar{f},\ \bar{z}=\text{conjugate}].\end{matrix}\right\} \quad (2)$$

Further,

$$f(z) \text{ is regular for } x>0 \qquad \ldots \ldots \quad (3)$$

when dealing with passive systems (*i. e.*, systems not containing any sources of energy). (3) is a consequence of the energy principle, and has been proved by Routh and, more recently, by Cauer [3]. By virtue of (3), (II.) can be written

$$h(t)=\frac{1}{2\pi i}\lim_{\delta\to+0}\int_{\delta-i\infty}^{\delta+i\infty} e^{tz}\frac{f(z)}{z}\,dz, \qquad \ldots \quad \text{(II. } a\text{)}$$

and this implies (applying Cauchy's theorem)

$$h(t)\equiv 0 \quad \text{for} \quad t<0. \qquad \ldots \ldots \quad (4)$$

[3] W. Cauer, *Mathemat. Annalen*, cvi. p. 379 (1932).

(3) and, therefore, (II. *a*) and (4) are assumed from now on. But it should be remembered that this does not yet imply the restriction to passive systems, as (3) is necessary, but not sufficient, for passivity ; this is well known from theory [4] and practical applications [5].

If $f(z)$ is a driving-point function [6] of a passive network denoted by $f_d(z)$, the energy principle requires that its ohmic part is non-negative when applying sinusoidal frequencies,

$$u_d(iy) \geqq 0 \text{ for any } y. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(3) and (5) are necessary and sufficient conditions for passive driving-point functions [3], [4], [8].

When dealing with general characteristics $f(z)$, *e. g.*, with mutual impedances, the condition corresponding to (5) involves two allied driving-point functions f_{d_1} and f_{d_2} ; it runs

$$u_{d_1}(iy)u_{d_2}(iy) - [u(iy)]^2 \geqq 0 \text{ [7]}. \quad . \quad . \quad . \quad (6)$$

For proof see Cauer, *l. c.* [3] and Gewertz [8].

Since a potential function has its extrema on the border of a regularity domain, we have

$$u_d > 0 \quad \text{for} \quad x > 0 \text{ [9]} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

by virtue of (3) and (5)

When the equality sign holds in (5) at an imaginary point $\pm iy_r$, the condition (7) requires

$$u'_d(\pm iy_r) = \text{real and} > 0, \quad . \quad . \quad . \quad (8\,a)$$

if $f(z)$ is regular at y_r ; if $\pm iy_r$ is a pole, (7) requires

$$\pm iy_r = \text{pole of 1st order of } f_d \text{ with real positive residue [10]}. \quad . \quad . \quad . \quad (8\,b)$$

[4] W. Cauer, *l. c.*, and O. Brune, Journ. Math. Phys. Massach. Inst. Technol. 1931, p. 191.

[5] Nyquist, Bell. Syst. Techn. Journ. xi. p. 126 (1933). Peterson, Kreer, and Ware, Proc. I. R. E. xxii. p. 1191 (1934).

[6] *E. g.*, a driving-point impedance ; more generally an image function with the two special properties : (1) it connects two variables which belong to the same element or terminal pair of the network ; (2) it reduces to a constant when all reactive elements are eliminated from the network. If an image function fulfils the first, but not the second, condition, it is usually possible to turn it into a driving-point function by multiplying it by a suitable power of z.

[7] If $f(z)$ is not an impedance or admittance or a related quantity, it is usually to be multiplied by a certain power of z before applying, (6) ; see [6].

[8] C. M. Gewertz, ' Network Synthesis,' part xi. (London, 1933).

[9] O. Brune, *l. c.* [4] W. Cauer, *l. c.* [3].

[10] R. Foster, Bell Syst. Techn. Journ. iii. p. 259 (1924).

[If there is an infinite number of such poles, and they are dense, *i. e.*, the y-axis or a part of it cannot be passed by analytic continuation, then it can be shown that

Sum of residues of imaginary poles of $f_d(z)$=finite (the set of poles is denumerable [11]). . . . (8 *c*).]

If $f(z)$ belongs to a network with a finite number of lumped elements, it is well known that

$f(z)$ is a rational function. . . . (9)

[If $f(z)$ is not a driving point function, it follows from (6) that, instead of (8 *b*), (8 *c*).

$\pm iy_r$ = pole of 1st order of $f(z)$ with real residue ; . . . . . (8 *d*)

Sum of the absolute values of the residues of imaginary poles of $f(z)$=finite. . . . (8 *e*).]

The general analytic properties of $f(z)$ thus summarized allow us to write the Fourier integral (II. *a*) in real form ; this will be useful, as relations between real quantities are required. The transformation is possible because of (2), and because in the right half-plane $f(z)$ can be written in terms of its values at the positive imaginary axis (positive frequencies).

Let us first suppose that $f(z)$ has only a finite number of poles at the imaginary axis. Then

$$f(z)=f_p(z)+a_\infty z+\frac{a_0}{z}+\sum_{(r)}\frac{2a_r z}{z^2+y_r^2}, \quad . \quad . \quad (10)$$

where

$f_p(iy)$ is regular for every y_1 and $\left\{\begin{matrix} a_0 \\ a_\infty \\ a_r \end{matrix}\right\}$ = real = positive

if $f(z)=f_d(z)$.

The second part being completely described by the poles y_r and their residues a_r, it remains to write $f_p(z)$ in terms of its values for $z=iy$. Dropping again the index p, we have

$$f(z)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{f(\zeta)}{z-\zeta}d\zeta; \quad 0=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{f(\zeta)}{z+\zeta}d\zeta;$$

$$\zeta=\xi+i\eta; \quad x\geqslant 0.$$

[11] G. C. Evans, "The Logarithmic Potential," chapters ii. and iii. Amer. Math. Soc. Colloqu. Publ. vol. vi. (New York, 1927).

When adding and subtracting the two equations and accounting for (2), two equivalent forms of the Poisson integral,

$$f(z)=\frac{2}{\pi}z\int_0^\infty \frac{u(i\eta)}{\eta^2+z^2}d\eta=f(\infty)-\frac{2}{\pi}\int_0^\infty \frac{\eta v(i\eta)}{\eta^2+z^2}d\eta, \quad (11\,a,\,b)$$

are obtained. Similarly, the well-known real forms of the Fourier integral

$$h(t)=\frac{2}{\pi}\int_0^\infty \frac{u(i\eta)}{\eta}\sin t\eta\, d\eta=u(0)+\frac{2}{\pi}\int_0^\infty \frac{v(i\eta)}{\eta}\cos t\eta\, d\eta;\quad t\geqslant 0, \quad (12\,a,\,b)$$

are obtained when applying (II. a) to both $t>0$ and $t<0$ and accounting for (4). (12), and all the conclusions derived from it in the following, are valid for $t\geqslant 0$ only; this will not be mentioned specially from now on.

The forms (11 a), (12 a) will be used throughout the following, although similar results may be derived from (12 b). [$u(i\eta)$ and $v(i\eta)$ are connected with each other by (11).] The former integrals have the advantage of being monotonic for driving-point functions because of (5); further $u(iy+\delta)$ behaves for $\delta\to+0$ at poles iy_r as an impulse function (see (1)),

$$D_0'(x;\,x_0)\equiv D_1(x;\,x_0)=\begin{cases}0, & x\neq x_0\\ \infty, & x=x_0\end{cases};\quad \int_{x_0-\epsilon}^{x_0+\epsilon} D_1\,dx=1, \quad (13)$$

by virtue of (8), while v oscillates. Therefore singularities at the imaginary axis can be allowed for in (11 a), (12 a) by simply using Stieltjes integrals,

$$\left.\begin{aligned} f(z)&=a_\infty z+\frac{a_0}{z}+\frac{2}{\pi}z\int_{+0}^\infty \frac{dU(\eta)}{\eta^2+z^2}=a_\infty z+\frac{2}{\pi}z\int_0^\infty \frac{dU(\eta)}{\eta^2+z^2}\\ &\text{with}\\ U(\eta)&=\int_{+0}^{\eta} u(i\rho)d\rho \quad\text{or}\quad =\int_0^\eta u(i\rho)d\rho, \text{ respectively;}\end{aligned}\right\} \quad (14)$$

correspondingly

$$\begin{aligned} h(t)&=a_\infty D_1(t,0)+a_0 t+\frac{2}{\pi}\int_{+0}^\infty \frac{\sin t\eta}{\eta}dU(\eta)\\ &=a_\infty D_1(t,0)+\frac{2}{\pi}\int_0^\infty \frac{\sin t\eta}{\eta}dU(\eta). \end{aligned} \quad (15)$$

$+0$ and $0\equiv\frac{1}{2}\{(+0)+(-0)\}$ denote that the interval of integration is open or closed, respectively. The

function $U(\eta)$ is bounded and non-decreasing (discontinuously increasing at the imaginary poles of f only) if $f(z)$ is a function $f_d(z)$; if not, $U(\eta)$ is of limited variation (*i. e.*, difference of two monotonic functions). The Poisson-Stieltjes integral representation and condition (3), or (3) and (5) respectively, imply each other ; for proof see Evans, *l. c.* [11] ; further [12].

For the sake of brevity (11 *a*) and (12 *a*) will be used instead of the more complete equations (14) and (15) [13], which may be resumed at any step. Introducing the conventional notations ω instead of η and $r(\omega)$ for the ohmic part $u(i\eta)$, they are rewritten

$$f(z)=\frac{2}{\pi}z\int_0^\infty \frac{r(\omega)}{\omega^2+z^2}d\omega\,; \quad h(t)=\frac{2}{\pi}\int_0^\infty \frac{r(\omega)}{\omega}\sin t\omega\, d\omega. \qquad . \; . \; . \quad (16\,a, b)$$

Some of the following results, though of general validity, bear little significance for transfer functions of highly dispersive systems for which $r(\omega)$ is rapidly oscillating in those frequency ranges where $|r|$ is not negligibly small. In such cases relations between steady state and transient errors are conveniently obtained by using the complex representation (II. *a*) and applying the integration method of steepest descent. For a general treatment of the signal propagation in highly dispersive systems see [14].

Excluding such cases for the moment, a mere dimensional consideration shows that deviations between two functions f_1 and f_2 at low frequencies will correspond to deviations of the associated functions $h_1(t)$ and $h_2(t)$ at late times t, while deviations at high frequencies will cause a different fine structure of h_1 and h_2. These vague statements become a little more precise if we consider the results of term-by-term integration of the power expansions in z or z^{-1} of (I.) or (II. *a*), respectively and compare them for the indices 1 and 2. But for well-known reasons those expansions have only a limited value. Some simple results of a more general kind are derived in the following.

[12] W. Cauer, Bell. Amer. Math. Soc., October 1932, p. 713.

[13] This is well justified by the consideration that some dissipation is always present in physical systems.

[14] H. G. Baerwald, *Annalen d. Physik*, vi. p. 295 (1930) ; vii. p. 731 (1930) ; viii. p. 565 (1931).

Let $f_1(z)$ and $f_2(z)$ be two image functions regular at $x>0$ which fulfil the condition

$$|r_1(\omega)-r_2(\omega)| \leqslant \delta_0 \quad \text{for} \quad \omega \geqslant \omega_0. \quad . \quad . \quad (17)$$

Then it follows that

$$|h_1(t)-h_2(t)| < \delta_0 + \frac{2}{\pi}$$

$$\times \left[\frac{1}{\omega_0}\left|\int^{\omega_0}\{r_1(\omega)-r_2(\omega)\}d\omega\right| - \delta_0\right]\omega_0 t \text{ for } \omega_0 t \lesssim 1. \quad (18)$$

If f_1 and f_2 are driving-point functions with the mutual relation

$$\left|\frac{r_1(\omega)-r_2(\omega)}{r_1(\omega)}\right| \leqslant \Delta_0 \text{ for } \omega \geqslant \omega_0, \quad . \quad . \quad (17\,a)$$

then

$$\frac{|h_1(t)-h_2(t)|}{h_1(t)} < \Delta_0 + \frac{2}{\pi}\left[\frac{1}{\omega_0}\int_0^{\omega_0}\{r_1(\omega)-r_2(\omega)\}d\omega\right.$$

$$\left. - \frac{\Delta_0}{\omega_0}\int_0^{\omega_0} r_1(\omega)d\omega\right]\frac{\omega_0 t}{h_1(t)} \text{ for } \omega_0 t \lesssim 1. \quad (18\,a)$$

The proof is very simple:

$$h_1(t)-h_2(t) = \frac{2}{\pi}\left(\int_0^{\omega_0} + \int_{\omega_0}^{\infty}\right)\{r_1(\omega)-r_2(\omega)\}\frac{\sin t\omega}{\omega}d\omega,$$

$$|h_1(t)-h_2(t)| < \delta_0\left\{1-\frac{2}{\pi}\int_0^{\omega_0}\frac{\sin t\omega}{\omega}d\omega\right\}$$

$$+\frac{2}{\pi}\int_0^{\omega_0}\{r_1(\omega)-r_2(\omega)\}\frac{\sin t\omega}{\omega}d\omega \text{ by virtue of (17).}$$

Since $t\omega_0 \lesssim 1$, $\sin t\omega \simeq t\omega$ for $\omega < \omega_0$. This gives (18) at once. When starting from (17 *a*) the proof is exactly similar when taking (5) into account.

The term const. $(\omega_0 t)$ represents the beginning of a power expansion whose continuation, however, would not bring much practical advantage in view of the estimating character of the formula and of the information (17). The result may prove helpful when transients are repeatedly, *e. g.*, periodically, started before approaching the steady state, and when, in addition to the main information (17) or (17 *a*), data are sufficient for estimating the mean value of (r_1-r_2) in the interval $(0, \omega_0)$.

We consider now the complementary case,

$$|r_1(\omega)-r_2(\omega)| \leqslant \delta_0 \quad \text{for} \quad \omega \leqslant \omega_0. \quad . \quad . \quad (19)$$

As $h_1(t)$ and $h_2(t)$ will mainly differ in fine structure and at early times, it will be less promising to compare them directly than to study relations between corresponding mean-value functions $\bar{h}_1(t\,;\,t_0)$, $\bar{h}_2(t\,;\,t_0)$ derived by an operation

$$\bar{h}(t\,;\,t_0)=\frac{1}{t_0}\int_{-\infty}^{+\infty}\mu\Big(\frac{t-\tau}{t_0}\Big)h(\tau)\,d\tau \quad . \quad . \quad (20\,a)$$

with an even and non-negative momentum function $\mu(x)$, which may be assumed as normalized,

$$\mu(-x)=\mu(x), \quad \mu(x)\geqq 0, \quad \int_{-\infty}^{+\infty}\mu(x)\,dx=1, \quad (20\,b)$$

while the time-constant t_0 is reciprocal to ω_0,

$$\alpha_0 \equiv \omega_0 t_0=\text{const.}, \quad . \quad . \quad . \quad . \quad . \quad (20\,c)$$

and gives the length of the interval of the mean-value process.

A Fourier integral representation (16) for $\bar{h}(t\,;\,t_0)$ is obtained similarly to the product rule due to Borel ("produit de composition" [15]) which states, when using the notation (III.):

supposing $f_1(z)\doteqdot h_1(t)$ and $f_2(z)\doteqdot h_2(t)$,

$$\text{then} \quad f_1(z)f_2(z)\doteqdot \left\{\begin{array}{l} \dfrac{d}{dt}\displaystyle\int_0^t h_1(\tau)h_2(t-\tau)\,d\tau, \\ \dfrac{d}{dt}\displaystyle\int_0^t h_2(\tau)h_1(t-\tau)\,d\tau. \end{array}\right\} \quad . \quad . \quad . \quad (21)$$

[Considering that the definition of $h(t)$ yields the representation

$$S_2(t)=\frac{d}{dt}\int_0^t S_1(t-\tau)h(\tau)\,d\tau=\frac{d}{dt}\int_0^t h(t-\tau)S_1(\tau)\,d\tau, \quad (22\,a)$$

if $\qquad S_1(t)=0$ for $t>0$ corresponding to (4), $\quad . \quad (22\,b)$

while the interpretation of the images

$$F_1(z)\doteqdot S_1(t) \quad \text{and} \quad F_2(z)\doteqdot S_2(t)$$

as signal spectra of S_1 and S_2 gives the alternative representation

$$S_2(t)\doteqdot F_2(z)=F_1(z)f(z)\,; \quad f(z)\doteqdot h(t), \quad . \quad (22\,c)$$

(21) is an immediate consequence of the superposition principle [16].]

[15] Borel, 'Leçons sur les Séries Divergentes,' p. 104 (Paris 1901).
[16] See B. v. d. Pol, *l. c.* [1], the last of his publications cited.

By virtue of (4) and of the restriction following (12), (20 *a*) can be written

$$\bar{h}(t\,;\,t_0)=\frac{1}{t_0}\int_0 \mu\left(\frac{t-\tau}{t_0}\right)h(\tau)\,d\tau. \quad . \quad . \quad (20\,d)$$

Applying here II. *a* and I., and defining the image function $m(z)$ by

$$\mu(t)\doteqdot z\,.\,m(z), \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

we obtain directly

$$\bar{h}(t\,;\,t_0)\doteqdot m(t_0 z)\,.\,f(z)\,; \quad . \quad . \quad . \quad . \quad (24)$$

or, when using the real form (16 *b*) and taking (20 *b*) into account,

$$\bar{h}(t\,;\,t_0)=\frac{2}{\pi}\int_0^\infty\left[2\int_0^\infty \mu(x)\cos\,(t_0\,\omega x)\,dx\right]r(\omega)\,\frac{\sin t\omega}{\omega}\,d\omega.$$

$$. \quad . \quad . \quad (25)$$

(The assumption is made that changing the order of integration is permissible.)

The special choice of $\mu(x)$ will depend on numerical considerations. Concluding, in analogy to (17), (18),

$$\left.\begin{aligned}&|\bar{h}_1(t\,;\,t_0)-h_2(t\,;\,t_0)|\\ &\quad<\frac{2}{\pi}\delta_0\int_0^{\omega_0}\left[2\int_0^\infty \mu(x)\cos\,(t_0\omega x)\,dx\right]\frac{\sin t\omega}{\omega}\,d\omega\\ &\quad+\frac{2}{\pi}\left|\int_{\omega_0}^\infty\left[2\int_0^\infty \mu(x)\,\cos\,(t_0\omega x)\,dx\right]\right.\\ &\qquad\qquad\left.\times\{r_1(\omega)-r_2(\omega)\}\,\frac{\sin t\omega}{\omega}\,dw\right|,\end{aligned}\right\}. \quad (26)$$

suitable estimations of the two right-side (r.s.) terms are wanted ; they should be neither too involved nor too inflexible for adaptation to the information about (r_1-r_2) available besides (19). The numerical value of the constant α_0 (20 *c*) will also depend on the choice of $\mu(x)$; but we may assume

$$\alpha_0 \text{ of the order } 2\pi_1, \quad . \quad . \quad . \quad . \quad . \quad (27)$$

i. e., the mean-value time roughly equal to the period $\frac{2\pi}{\omega_0}$. For much smaller α_0 the application of the mean-value process (20) cannot have a noticeable effect on the second term of (26) ; on the other hand, assuming α_0

much larger would mean wasting the information (19). It can further be supposed that

$$t \gtrsim t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

because the reduction of the fine structure of h by applying (20) makes statements " empty " for $t<t_0$.

The simplest $\mu(x)$ lending itself is the " rectangular function "

$$\mu^{(0)}(x)=\left\{\begin{matrix}1 & \text{for} & |x|<\frac{1}{2}\\ 0 & \text{for} & |x|>\frac{1}{2}\end{matrix}\right\} \quad . \quad . \quad . \quad . \quad (29)$$

Though it will prove unsuited for application in (26), it is important as basic function for defining the length of the mean-value interval. This is seen when considering any other function $\mu(x)$ which is compatible with (20 *b*). Since

$$\mu(x\,;\,q)\equiv q\,.\,\mu(qx) \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

with the arbitrary time-scale constant q fulfils (20 *b*) as well, the results of applying different momentum functions are comparable only after defining q, thus normalizing the " width " besides the area (20 *b*). This definition is most suitably based on $\mu^{(0)}(x)$ with $q^{(0)}=1$ by fixing

$$\left.\begin{aligned}&\int_0^{\infty}\{\mu^{(0)}(x)-\mu(x\,;\,q)\}^2\,dx=\min.(q),\\ &\text{or}\quad \mu\left(\frac{q}{2}\right)=\int_0^{\infty}[\mu(x)]^2\,dx.\end{aligned}\right\} \quad . \quad . \quad (31)$$

The value q thus implicitly determined is called q_0. Then writing

$$m(\xi)=2\int_0^{\infty}\mu(x)\cos\xi x\,dx. \quad . \quad . \quad . \quad (32)$$

for the Fourier transform of $\mu(x)$ in consequence of (23) and (20 *b*), and defining

$$\theta=\frac{t}{t_0}, \quad \Omega=\frac{\omega}{\omega_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

(25), (26) run

$$h(t_0\theta\,;\,t_0)=\frac{2}{\pi}\int_0^{\infty} r(\omega_0\Omega)\,.\,m\left(\frac{\alpha_0}{q_0}\Omega\right)\frac{\sin\theta\Omega}{\Omega}\,d\Omega, \quad . \quad (34)$$

$$|\bar{h}_1(t_0\theta\,;\,t_0)-\bar{h}_2(t_0\theta\,;\,t_0)|<\frac{2}{\pi}\delta_0\int_0^1 m\left(\frac{\alpha_0}{q_0}\Omega\right)\frac{\sin\theta\Omega}{\Omega}\,d\Omega$$

$$+\frac{2}{\pi}\left|\int_1^{\infty}\{r_1(\omega_0\Omega)-r_2(\omega_0\Omega)\}m\left(\frac{\alpha_0}{q_0}\Omega\right)\frac{\sin\theta\Omega}{\Omega}\,d\Omega\right|. \quad (35)$$

Some pairs,

$$q_0 \cdot \mu(q_0 x) \; ; \quad m\left(\frac{\alpha_0}{q_0}\Omega\right),$$

Fig. 1.

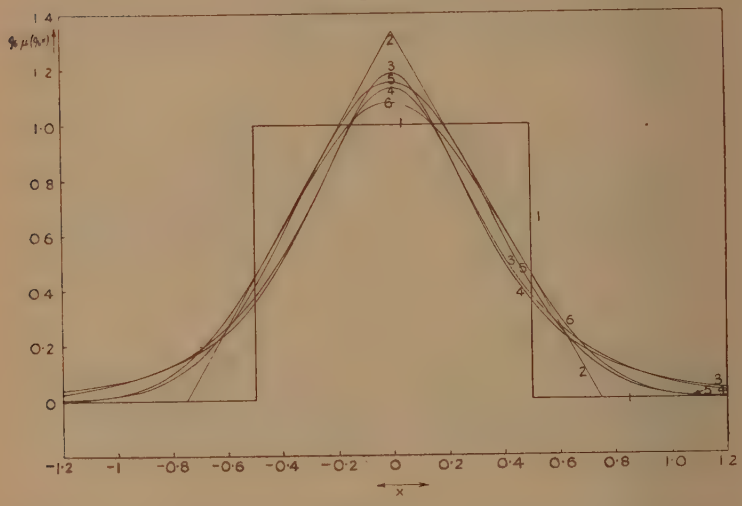
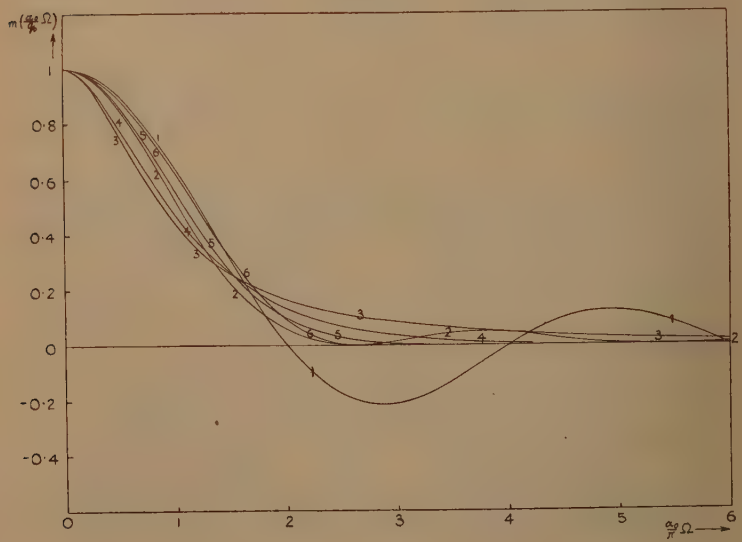
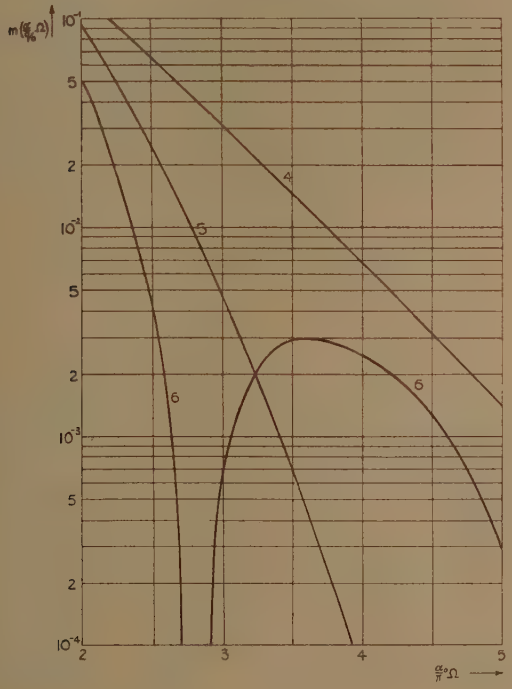


Fig. 2.



are now considered from which a most suitable momentum function will be selected. For the sake of brevity we write simply $\mu(x)$ and $m(\xi)$ according to (32). In some particular cases, q_0 (31) will be given numerically. Corresponding momentum functions $q_0 \cdot \mu(q_0 x)$ and their Fourier transforms are plotted in figs. 1, 2, and 3 respectively.

Fig. 3.



The basic pair

$$\mu(x)=\mu^{(0)}(x)\,; \quad m^{(0)}(\xi)=\frac{\sin \frac{1}{2}\xi}{\frac{1}{2}\xi}\,; \quad q^{(0)}=1 \quad . \quad (36)$$

(curve 1) is not suited for application because $m^{(0)}(\xi)$ decreases slowly and in an oscillatory manner and, therefore, does not allow a rapid estimation of the remainder term of (35). This is due to the discontinuity of μ at $\pm\frac{1}{2}$, which causes a kind of resonance effect. An improvement

is obtained by making $\mu(x)$ continuous, *e. g.*, by using the function of triangular shape,

$$\mu(x) = \left\{ \begin{matrix} 1-x\ ;\ x \leqslant 1 \\ 0\ \ ;\ x \geqslant 1 \end{matrix} \right\}\ ;\quad m(\xi) = \left[\frac{\sin \frac{1}{2}\xi}{\frac{1}{2}\xi} \right]^2$$

$$= [m^{(0)}(\xi)]^2\ ;\ q_0 = \frac{4}{3}, \quad . \quad . \quad . \quad (37)$$

which is plotted as curve " 2 " in figs. 1 and 2. More generally, when choosing $\mu(x)$ as polynomial of nth degree for $|x| \leqslant 1$ and $\mu \equiv 0$ for $|x| \geqslant 1$, and keeping all the derivatives of μ up to the nth continuous at ± 1, m decreases finally as $\xi^{-(n+1)}$ multiplied by an oscillatory term ; this follows from (32) after n partial integrations.

The same asymptotic behaviour of $m(\xi)$, but without the oscillatory term, ensues when, instead of breaking up a polynomial $\mu(x)$ at ± 1, it is multiplied by the exponential term $e^{-|x|}$. In the corresponding pairs,

$$\left.\begin{aligned} \mu(x) &= \frac{1}{2} \sum_{\nu=0}^{n} \frac{a_\nu}{\nu!} |x|^\nu e^{-|x|}\ ;\quad \sum_{\nu=0}^{n} a_\nu = 1, \\ m(\xi) &= \sum_{\nu=0}^{n} a_\nu \mathrm{Re}\{(1-i\xi)^{-(\nu+1)}\}, \end{aligned}\right\} \quad . \quad . \quad (38)$$

$m(\xi)$ is a rational function with the denominator $(1+\xi^2)^{n+1}$. The degree of the numerator which, in general, is $2n$, can be lowered by choosing the ratios $\frac{a_\nu}{a_0}$ suitably, and finally becomes zero. The corresponding pairs are

$$\mu(x) = \frac{e^{-|x|}}{e^{2n+1}} \sum_{\nu=0}^{n} \frac{(2n-\nu)!}{(n-\nu)!\, n!\, \nu!} ((2|x|)^\nu ;\ m(\xi) = (1+\xi^2)^{-(n+1)}.$$

. . . (39)

The curve " 3 " of figs. 1 and 2 belongs to the pair with $n=1$:

$$q_0 = 4{\cdot}77_4. \quad . \quad . \quad . \quad . \quad . \quad . \quad (39\,a)$$

The expression (39) for general n is conveniently obtained by reciprocating the Fourier transform (32) [17]. Since $\mu(x)$ is an even function, the same is true for $m(\xi)$, and it is simply

$$\mu(x) = \int_0^\infty m(\xi) \cos x\xi\, d\xi, \quad . \quad . \quad . \quad . \quad (40)$$

[17] See, *e. g.*, Courant-Hilbert, 2nd ed. vol. i. p. 68.

also in direct consequence of (23). It follows further from the third equation of (20 *b*) that

$$m(0)=1, \quad \ldots\ldots \quad (41)$$

while by virtue of (40)

$$\int_0^\infty m(\xi)\,d\xi=\mu(0). \quad \ldots\ldots \quad (42)$$

More rapidly decreasing functions $m(\xi)$ are obtained by reciprocating (38), (39), as then $m(\xi)$ takes on the factor $e^{-|\xi|}$:

$$\mu(x)=\frac{2^{2n}(n\,!)^2}{(2n)\,!\,\pi}(1+x^2)^{-(n+1)};$$

$$m(\xi)=e^{-\xi}\sum_{\nu=0}^{n}\frac{n\,!\,(2n-\nu)\,!}{(2n)\,!\,(n-\nu)\,!\,\nu\,!}(2\xi)^\nu. \quad \ldots \quad (43)$$

The pair for $n=1$,

$$\left.\begin{aligned}&\mu(x)=\frac{2}{\pi}(1+x^2)^{-2}; \quad m(\xi)=(1+\xi+\tfrac{1}{3}\xi^2)e^{-\xi};\\ &q_0=2\sqrt{\frac{4}{\sqrt{5}}-1}=1{\cdot}77_7,\end{aligned}\right\} \quad (44)$$

is plotted as curve " 4 " in figs. 1–3.

When proceeding to the limit $n\to\infty$ in (38), (39) functions $m(\xi)$ are obtained which decrease still more rapidly, viz., with $e^{-\xi^2}$. This is true because then the parameter q_0, which has been omitted in the preceding pairs, increases like n. The function $e^{-\xi^2}$ of the normal error law is identical with its Fourier transform. It shares this property with a series of more general functions, polynomials in ξ^2 multiplied by $e^{-\xi^2}$, which are closely related to the derivatives of the Gaussian error function (Gram-Charlier expansion) or the eigenfunctions of the parabolic cylinder [18]. As these functions oscillate, and, therefore, cannot serve as momentum functions, we consider compounds

$$\left.\begin{aligned}&\mu(x)=\sum_{\nu=0}^{n}\frac{a_\nu}{\pi(\nu-\tfrac{1}{2})}x^{2\nu}e^{-x^2}; \quad \sum_{\nu=1}^{n}a_\nu=1,\\ &m(\xi)=\sum_{\nu=0}^{n}(-1)^\nu\frac{\pi(-\tfrac{1}{2})}{\pi(\nu-\tfrac{1}{2})}a_\nu\frac{d^{2\nu}}{(d\xi)^{2\nu}}\left\{e^{-\left(\frac{\xi}{2}\right)^2}\right\}.\end{aligned}\right\} \quad (45)$$

[18] See, *e. g.*, G. A. Campbell and R. M. Foster, " Fourier Integrals for Practical Applications," Bell Teleph. Syst. Monograph, B. 584, Sept. 1931.

Curve " 5 " of figs. 1–3 gives the fundamental

$$\sqrt{\pi}\mu(x)=e^{-x^2}\,;\quad m(\xi)=e^{-\left(\frac{\xi}{2}\right)^2}\,;\quad q_0=2{\cdot}03_8. \quad . \quad (46)$$

This function pair is better suited for application as momentum function than any of those preceding (45) because it combines rapid and monotonic asymptotic decrease with analytic simplicity. When wanting to combine these advantages of $m(\xi)$ with a closer approximation of the rectangular shape by $\mu(x)$ one may try a more general series (45) with the a_ν so chosen that $m(\xi)$ does not become negative. This process, however, does not lead to an improvement. It is sufficient to consider the simplest pair,

$$\left.\begin{aligned}\sqrt{\pi}\mu(x)&=\left(\frac{3}{4}+\frac{x^2}{3}+\frac{x^4}{9}\right)e^{-x^2}\,;\\ m(\xi)&=\left\{1-\frac{1}{3}\left(\frac{\xi}{2}\right)^2\right\}e^{-\left(\frac{\xi}{2}\right)^2}\,;\quad q_0=2{\cdot}55_2,\end{aligned}\right\}. \quad (47)$$

plotted as curve " 6 ". Fig. 1 shows that the improvement over " 5 " is only slight. There is also no appreciable difference between the corresponding curves of fig. 2 in the region $\frac{\alpha_0}{\pi}\Omega=2$, which according to (27) corresponds to the frequency ω_0. For $\frac{\alpha_0}{\pi}\Omega>2$ curve " 6 " starts decreasing more rapidly owing to the double zero at $\xi=2\sqrt{3}$, but the contrary is true at higher values (see the logarithmic representation, fig. 3). Thus it will depend on how rapidly (r_1-r_2) increases above ω_0 in a special case whether (46) or a function of the kind (47) is preferred. For a theorem of general significance, however, the simple pair (46) is definitely superior, and will be used henceforth :

$$\bar{h}(t_0\theta\,;\,t_0)=\frac{q_0}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-q_0^2(\tau-\theta)^2}h(t_0\tau)d\tau$$

$$=\frac{2}{\pi}\int_0^\infty r(\omega_0\Omega)e^{-\left(\frac{\alpha_0}{2q_0}\right)^2\Omega^2}\,\frac{\sin\theta\Omega}{\Omega}d\Omega\,;\quad q_0=2{\cdot}03_8\,;$$

$$|\bar{h}_1(t_0\theta\,;\,t_0)-\bar{h}_2(t_0\theta\,;\,t_0)|<\frac{2}{\pi}\delta_0\int_0^1 e^{-\left(\frac{\alpha_0}{2q_0}\right)^2\Omega^2}\,\frac{\sin\theta\Omega}{\Omega}d\Omega$$

$$+\frac{2}{\pi}\left|\int_1^{\infty}\{r_1(\omega_0\Omega)-r_2(\omega_0\Omega)\}e^{-\left(\frac{\alpha_0}{2\mu_0}\right)^2\Omega}\frac{\sin\theta\Omega}{\Omega}d\Omega\right|. \quad . \quad . \quad . \quad (48\,a,\,b)$$

The next step will be the evaluation of the right side of this inequality by means of approximations which are sufficiently general to be applicable to most problems of this kind. The favourable properties of the Gaussian momentum function will prove decisively helpful for this purpose.

When either or both quantities

$$\beta_0 \equiv \frac{\alpha_0}{2q_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and θ increase, the two r.s. terms of (48 *b*) evidently tend to δ_0 and zero respectively ; the approach becomes rapid for larger values of these parameters [19]. This suggests approximations of an asymptotic character. Such approximations are most conveniently obtained by carrying integration back into the complex domain and proceeding along the " steepest descent " [20].

Before carrying out this calculation some definitions are required concerning the numerical values of β_0 and θ, and some assumptions must be made about the difference function

$$r_\Delta(\omega)\equiv r_1(\omega)-r_2(\omega), \quad . \quad . \quad . \quad . \quad (50)$$

for $\omega>\omega_0$. According to (28) we have

$$\theta \gtrsim 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (51\,a)$$

On the other hand, $\alpha_0=2\pi$ in (27) leads to $\beta_0=1{\cdot}55$ by (46). (27) is only an approximate relation ; for numerical application we will assume the interval

$$1\gtrsim\beta_0\gtrsim 2,\ 4\ (\theta \gg 1), \quad . \quad . \quad . \quad . \quad (51\,b)$$

which corresponds to a time-range

$$t_0\sim(0{\cdot}64_5 \quad \text{to} \quad 1{\cdot}55).\frac{2\pi}{\omega_0} \quad . \quad . \quad . \quad (51\,c)$$

[19] This is due to the rapid decline of the higher frequency components when increasing t_0 beyond $\frac{2\pi}{\omega_0}$, and to the attenuation factors of the natural oscillations of the system respectively.

[20] See, *e. g.*, Courant-Hilbert, vol. i. 2nd ed. (1931), p. 455 ; Franck-Mises, vol. i. ed. 1925, p. 357 ; or P. Debye, *Math. Annalen*, lxvii. p. 535 (1909).

of the equivalent rectangular momentum function. In general the value of β_0 will not be independent of θ; usually (viz. if dissipation is present) β_0 may tend to 0 for $\theta >> 1$, *i. e.*, the process (20) becomes insignificant. The physical explanation is obvious (see [19]).

The chief information supplied about $r_\Delta(\omega)$ is (19), and it can generally be assumed that

$$|r_\Delta(\omega_0)| = \delta_0 ; \left[\frac{r'_\Delta}{r_\Delta}\right]_{\omega=\omega_0} > 0. \quad . \quad . \quad . \quad (52)$$

Some secondary information is required concerning the domain $\omega > \omega_0$, which, however, might be comparatively vague. Its general form—as continuation of (19)—will be a majorant $r^*_\Delta(\omega)$, *i. e.*, a function with the property

$$|r^*_\Delta(\omega)| \geqslant |r_\Delta(\omega)| \quad \text{for} \quad \omega > \omega_0. \quad . \quad . \quad . \quad (53)$$

The " quality of information " depends on the reciprocal to the relative margin between r^*_Δ and the actual function r_Δ (which is not supposed to be known). This margin, vanishing for ω_0 (52), will increase with $(\omega - \omega_0)$. As no information is supposed to be available besides (19) and r^*_Δ, this function must be introduced into the inequality (48 *b*). It is convenient to relate r^*_Δ to δ_0 and ω_0 by defining

$$\rho(\Omega - 1) \equiv \frac{r^*_\Delta(\omega)}{\delta_0} \quad \text{for} \quad \omega \geqslant \omega_0 ; \ \rho(0) = 1, \ \rho'(0) > 0. \quad (54)$$

This leads to the final form

$$\frac{|\bar{h}_1(t_0\theta ; t_0) - \bar{h}_2(t_0\theta ; t_0)|}{\delta_0} < \frac{2}{\pi} \int_0^1 e^{-\beta_0^2\Omega^2} \frac{\sin \theta\Omega}{\Omega} d\Omega + \frac{2}{\pi} \left| \int_1^\infty \rho(\Omega - 1) e^{-\beta_0^2\Omega^2} \frac{\sin \theta\Omega}{\Omega} d\Omega \right|. \quad (55)$$

A standard expression will be assumed for $\rho(x)$ with the numerical values of its coefficients depending on the special case. $\rho(x)$ rises monotonically, or at least does not fluctuate too much up to $x \sim 1$, a polynomial

$$1 + \rho_1 x + \rho_2 x^2 + \ldots . ; \quad \rho_1 > 0 \quad . \quad . \quad (56a)$$

of a low degree is suited. High accuracy of matching a given information $\rho(x)$ is required for small values of x only, because of the influence of the exponential and oscillatory term in (55) for larger x apart from the growing

vagueness of information. (56 *a*) is unsuited only if $\rho(x)$ fluctuates rapidly in the important x-range, *i. e.*, has there sharp resonance peaks. They correspond to pole pairs of the functions $f_1(z)$ or $f_2(z)$ near the imaginary axis not far above ω_0. Calling $\delta_0\omega_0 a_\kappa$ the residue of a pole pair at $(\pm i\omega_\kappa - \delta_\kappa)$ with

$$\Omega_\kappa = 1 + x_\kappa = \frac{\omega_\kappa}{\omega_0} > 1, \quad \text{but not} \quad >> 1, \quad \frac{\delta_\kappa}{\omega_\kappa} << 1, \qquad (56\,b)$$

the corresponding partial fraction term is approximately

$$\frac{2\delta_0\omega_0 a_\kappa (z+\delta_\kappa)}{(z+\delta_\kappa)^2+\omega_\kappa^2}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56\,c)$$

which follows from (10) with $f_p(z)=0$ when shifting the y-axis by δ_κ. This gives a corresponding term of $\rho(x)$

$$\frac{a_\kappa d_\kappa}{d_\kappa^2+(x-x_\kappa)^2}; \quad d_\kappa \equiv \frac{\delta_\kappa}{\omega_0} << 1. \quad . \quad . \qquad (56\,d)$$

(56 *c*, *d*) is supposed to represent a " dominant " resonance *i. e.*,

$$\left[\left|\frac{d_\kappa}{a_\kappa}\rho(x_\kappa)\right| - 1\right]^2 << 1, \quad . \quad . \quad . \qquad (56\,e)$$

by (at least) one order of magnitude, for otherwise (56 *a*) will match the peak without requiring an unduly large number of terms. On the other hand, a term (56 *b*) may be neglected at frequencies sufficiently far from its resonance, *i. e.*, where

$$\left[\frac{d_\kappa}{a_\kappa}\rho(x)\right]^2 >> \left[1+\left(\frac{x-x_\kappa}{d_\kappa}\right)^2\right]^{-2} \sim \left[\frac{d_\kappa}{x-x_\kappa}\right]^4, \qquad (56\,f)$$

by (at least) one order of magnitude. It follows for the essential range of a dominant resonance term

$$|x-x_\kappa| \leqslant N_\kappa d_\kappa \quad \text{with} \quad N_\kappa^2 >> 1, \qquad (56\,g)$$

by (at least) one order of magnitude. The condition (52) requires that the peak is sufficiently far above $\kappa=0$. Taking the preceding inequalities into account, it follows that

$$N_\kappa d_\kappa \lesssim x_\kappa \equiv \Omega_\kappa - 1. \quad . \quad . \quad . \quad . \qquad (56\,h)$$

(56 *a*) and 56 *d*) lead to the standard form

$$\rho(x) = p(1+\rho_1 x+\rho_2 x^2+\ldots) + \sum_{(\kappa)} \frac{a_\kappa d_\kappa}{d_\kappa^2+(x-x_\kappa)^2}$$
$$\equiv \rho_P(x) + \sum_{(\kappa)} \rho_\kappa(x); \quad \rho(0)=1. \quad . \quad . \quad . \qquad (57)$$

It will be found suitable to match the given information in any practical case with a few terms only.

The first r.s. term of (55) is

$$\frac{2}{\pi}\int_0^1 e^{-\beta_0^2\Omega^2}\frac{\sin\theta\Omega}{\Omega}d\Omega=\Phi\left(\frac{\theta}{2\beta_0}\right)-\frac{2}{\pi}\int_{\beta_0}^{\infty} e^{-x^2}\sin\frac{\theta}{\beta}\frac{dx}{x}; \quad . \quad . \quad . \quad (58)$$

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt=\text{Gaussian error function} \quad (\text{notation of Jahnke-Emde}). \quad (59)$$

The r.s. integral of (58) may be written

$$\frac{2}{\pi}Im\left\{e^{-\beta_0^2+i\theta}\int_0^{\infty} e^{-[(2\beta_0^2+1-i\theta_0)\zeta+\beta_0^2\zeta^2]}\cdot\frac{e^{\zeta}}{1+\zeta}d\zeta\right\}. \quad (60\,a)$$

The value of (60 *a*) is not altered by a deformation of the path of integration which avoids the pole $|\zeta|=-1$ and leads towards infinity under an angle $\phi\geqq\frac{\pi}{4}$ and $>-\frac{\pi}{4}$.

Near $|\zeta|=0$ the absolute variation of the first factor of the integrand is large compared with that of the second factor when defining the variation V of an analytic function $w(z)$ by

$$V(z)=z\frac{w'(z)}{w(z)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Therefore we choose the path of maximum decrease of the first factor or "primary function." It is obtained by making its exponent negative real. The result is the hyperbola branch

$$\eta=\frac{\theta\xi}{2\beta_0^2(1+\xi)+1}; \quad (\zeta=\xi+i\eta\,;\ \xi,\eta\geqq 0), \quad . \quad (62)$$

which passes the origin in the direction

$$\text{tg}\,\phi_0=\frac{\theta}{2\beta_0^2+1} \quad . \quad . \quad . \quad . \quad (62\,a)$$

and has the asymptote

$$\eta=\frac{\theta}{2\beta_0^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (62\,b)$$

(see fig. 4 for illustration). It will be proved that with (51 *a*, *b*) integration may be discontinued at $|\zeta|=1$ (point A in fig. 4, convergence limit of the secondary factor of (60 *a*)), the remainder being negligible. This is even true when, instead of proceeding along the hyperbola,

the zero tangent is chosen as path of integration (point B ; broken line indicates remainder integral). This is preferable because otherwise integration does not lead to tabulated functions, and it is justified by the confluence of the two ways at the origin ; it may be interpreted by regarding $e^{-(2\beta_0{}^2+1-i\theta)\zeta}$ as primary and $\frac{e^{-\beta_0{}^2\zeta^2+\zeta}}{1+\zeta}$ as secondary component [21].

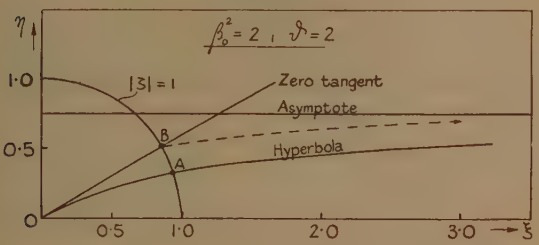
The integral (60 *a*) with the limits 0 and $e^{i\phi_0}$,

$$\frac{2}{\pi} Im \left\{ e^{-\beta_0{}^2+i(\theta+\phi_0)} \int_0^1 e^{-\sqrt{(2\beta_0{}^2+1)^2+\theta^2}\,.\,u} \frac{e^{e^{i\phi_0}u-\beta_0{}^2 e^{2i\phi_0}u^2}}{1+e^{i\phi_0}u}\, du \right\}, \quad . \quad . \quad . \quad (60\,b)$$

can be evaluated by expanding the secondary factor. Using the abbreviations

$$B_0 \equiv \sqrt{(2\beta_0{}^2+1)^2+\theta^2} \quad . \quad . \quad . \quad . \quad (63)$$

Fig. 4.



and

$$e_n(x) \equiv \frac{n!}{x^n}\left\{1-e^{-x}\sum_{\nu=0}^{n-1}\frac{x^\nu}{\nu!}\right\} = \sum_{\nu=0}^{\infty}\frac{\sum_{\rho=0}^{\nu}(-1)^\rho\binom{n+\nu}{\rho}}{\prod_{\rho=1}^{\nu}(n+\rho)}$$

$$= 1-\frac{x}{1+\frac{1}{n}}+\frac{1-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}\frac{x^2}{2!} - \frac{1+\frac{3}{n}+\frac{2}{n^2}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}\frac{x^3}{3!}+\ldots, \quad (64)$$

[21] The absolute variation of $e^{-(2\beta_0{}^2+1-i\theta)\zeta}$ is $|2\beta_0{}^2+1-i\theta|\,.\,|\zeta$, that of $e^{-\beta_0{}^2\zeta^2}$ is $2\beta_0{}^2|\zeta|^2$, *i. e.*, comparatively small for $|\zeta|<1$. See also Baerwald, *l. c.* [14], part 1, para. 11.

the result is

$$\frac{2}{\pi} e^{-\beta_0 x}\{e_1(B_0).\sin(\theta+\phi_0)-\tfrac{1}{2}(\beta_0^2-\tfrac{1}{2})\,e_3(B_0).\sin(\theta+3\phi_0)-\tfrac{1}{9}\,e_4(B_0).\sin(\theta+4\phi_0)+\tfrac{1}{8}(\beta_0^2-\tfrac{1}{2})^2\,e_5(B_0).\sin(\theta+5\phi_0)\ldots\} \quad . \quad . \quad . \quad (65)$$

In many cases the first term of this expansion

$$\frac{2}{\pi} e^{-\beta_0 x}\frac{(2\beta_0^2+1)\sin\theta+\theta\cos\theta}{(2\beta_0^2+1)^2+\theta^2}\{1-e^{-\sqrt{(2\beta_0^2+1)^2+\theta^2}}\}; \quad (65\,a)$$

will be sufficient, as (65) is an asymptotic term itself compared with $\Phi\left(\frac{\theta}{2\beta_0}\right)$ in (58). For the values $\beta_0^2=\theta=2$ of fig. 4 (65 *a*) has the value $1{\cdot}88\,.\,10^{-2}$, while

$$\Phi\left(\frac{\theta}{2\beta_0}\right)=\Phi\left(\frac{1}{\sqrt{2}}\right)=0{\cdot}683.$$

It remains to estimate the relative amount of the integral from B to ∞ (fig. 4). It is easily shown that it is smaller than the absolute ratio of the integrand values at $\zeta=e^{i\varphi_0}$ and $\zeta=0$ [22]

$$\exp.\left\{-\frac{2\beta_0^2(2\beta_0^2+1)+\theta^2}{\sqrt{(2\beta_0^2+1)^2+\theta^2}}-\frac{(2\beta_0^2+1)^2-\theta^2}{(2\beta_0^2+1)^2+\theta^2}\right\}\Big/\sqrt{2\left(1+\frac{2\beta_0^2+1}{\sqrt{(2\beta_0^2+1)^2+\theta^2}}\right)} \quad . \quad (66)$$

For the smallest values permitted by (51), viz., $\beta_0=\theta_0=1$, this becomes $2{\cdot}5\,.\,10^{-2}$; for the values of fig. 4, $1{\cdot}37\,.\,10^{-3}$.

The method of evaluating the second r.s. term of (55) is very similar. The process is different for the two components of $\rho(x)$ in (57). It requires the variation of the polynomial part to be small compared with the variation of the combined exponential and sinusoidal terms in (55), while the variation of a resonance term is supposed to be large compared with that of the exponential term in the ranges (56 *g*). The last condition is usually automatically fulfilled. But it may happen that this is not true for the first condition if $r_1(\omega)$ and $r_2(\omega)$ diverge rapidly for $\omega>\omega_0$ or if the information about $\rho(x)$ is not

[22] The reason is the preponderance of the quadratic term in the exponent for $|\xi|>1$; therefore the integrand decreases more rapidly than along OB (fig. 4).

sufficient. Then the relative contribution of the frequencies above ω_0 to the transient error might become considerable in spite of the process (20 *a*), (48 *a*), *i. e.*, the ρ-term in (55) is no longer merely corrective. Then either the initial part of the integral must be evaluated numerically or graphically, until asymptotic integration can be started, or the value of β_0, *i. e.*, t_0, must be sufficiently increased.

Starting the integration of the ρ-term with the polynomial part (56 *a*), the linear term of $\rho_P(x)$ is joined with the primary, the higher terms with the secondary component, as in (60 *b*). The result is

$$\left.\begin{aligned} &\frac{2}{\pi} p e^{-\beta_\nu^2} \sum_{\nu=1}^{\infty} \frac{A_\nu}{\nu+1} e_\nu(C_0) \cdot \sin(\theta+\nu\psi_0), \\ &\qquad C_0 e^{i\psi_0} \equiv (2\beta_0^2+1-\rho_1)+i\theta \\ &\text{with} \\ &A_1=1, \quad A_2=0, \quad A_3=-(\beta_0^2-\tfrac{1}{2})^2-(\tfrac{1}{2}\rho_1^2-\rho_2), \\ &A_4=-\tfrac{1}{3}-(\tfrac{1}{3}\rho_1^3+\rho_1\rho_2-\rho_3), \quad A_5=\tfrac{1}{2}(\beta_0^2-\tfrac{1}{2})^2 \\ &\quad +(\tfrac{1}{2}\rho_1^2-\rho_2)\beta_0^2+(-\tfrac{1}{8}\rho_1^4+\tfrac{1}{2}\rho_1^2\rho_2 \\ &\qquad +\tfrac{1}{4}\rho_2^2-\rho_1\rho_3+\rho_4), \ \ldots \end{aligned}\right\} \quad (67)$$

For $\rho_1=\rho_2=\ldots=0$ and $p=1$, (67) reduces, of course, to (65). Similarly (66) is a majorant of the relative error of (67) when multiplied by

$$\frac{1}{p} \left| \rho_P(e^{i\psi_0}) \right| . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66\,a)$$

This factor shows that it might be necessary to enlarge an adopted value (51 *b*) of β_0 for using the asymptotic approximation; on the other hand, (66) shows that β_0 may tend to 0, *i. e.*, the process (20) may be dropped, if θ becomes sufficiently large.

In the case of a sharp resonance term $\rho_\kappa(x)$, the integration interval is given by (56 *g*):

$$\frac{2}{\pi} \frac{a_\kappa}{\Omega_\kappa} e^{-\beta_\kappa^2} \int_{-N_\kappa}^{+N_\kappa} \frac{\sin \theta\Omega_\kappa(1+\Delta_\kappa y)}{1+y^2} \frac{e^{-\beta_\kappa^2 \Delta_\kappa y(2+\Delta_\kappa y)}}{1+\Delta_\kappa y} dy, \quad (68)$$

with the abbreviations

$$\beta_\kappa \equiv \beta_0 \Omega_\kappa, \quad \Delta_\kappa \equiv \frac{d_\kappa}{\Omega_\kappa} = \frac{\delta_\kappa}{\omega_\kappa} \ (56\,b, c). \quad . \quad . \quad (69)$$

The second factor, consisting of the exponential and the pole term, can be regarded as secondary if its variation in $(\pm N_\kappa)$ is comparatively small, *i. e.*, if

$$\beta_\kappa^2 N_\kappa \Delta_\kappa < 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

In general this condition will be numerically fulfilled as a consequence of the inequalities (56 *e–h*) and of (51 *b*), unless $\Omega_\kappa >> 1$; then, however, the contribution of the resonance term to (55) will usually be negligible. As $\beta_\kappa > 1$, the exponential part represents the bulk variation of the secondary factor. It is consistent with (70) to replace it by the first two terms of its Taylor expansion in the denominator, while one term is sufficient for the pole part. This leads to the approximation

$$\frac{2}{\pi}\frac{a_\kappa}{\Omega_\kappa} e^{-\beta_\kappa^2} \int_{-N_\kappa}^{+N_\kappa} \frac{\sin \theta\Omega_\kappa(1+\Delta_\kappa y)}{1+y^2} \times \frac{1-\Delta_\kappa y}{1+2\beta_\kappa^2 \Delta_\kappa y+\beta_\kappa^2(2\beta_\kappa^2+1)\Delta_\kappa^2 y^2}\, dy.$$

When extending this integral to $\pm\infty$, the relative error is at most of the order $\frac{1}{2}N_\kappa^{-1}$, *i. e.*, 10^{-1}, by (56 *g*). The integral

$$\frac{2}{\pi}\frac{a_\kappa}{\Omega_\kappa} e^{-\beta_\kappa^2} Im \left\{ \int_{-\infty}^{+\infty} \frac{e^{i\theta\Omega_\kappa(1+\Delta_\kappa y)}}{1+y^2} \times \frac{1-\Delta_\kappa y}{1+2\beta_\kappa^2 \Delta_\kappa y+\beta_\kappa^2(2\beta_\kappa^2+1)\Delta_\kappa^2 y^2} \right\} dy$$

is equal to the residue integrals of the two poles with positive imaginary parts, $y=i$ and

$$\Delta_\kappa y = \frac{-1+i\sqrt{1+\beta_\kappa^{-2}}}{2\beta_\kappa^2+1}.$$

The first one gives the main term. It is easily seen that the original integrand (68) may be reintroduced in it, within the above limits of error, but this is not allowed for the second residue. The result is

$$\frac{2a_\kappa}{\Omega_\kappa} e^{-\beta_\kappa^2}\left[e^{-\Delta_\kappa\Omega_\kappa\theta+(\beta_\kappa\Delta_\kappa)^2}.\sin\{\Omega_\kappa\theta-(2\beta_\kappa^2+1)\Delta_\kappa\}\right.$$

$$\left.-\Delta_\kappa\ \mathrm{F}_\kappa\ e^{-\frac{\sqrt{1+\beta_\kappa^{-2}}}{2\beta_\kappa^2+1}\Omega_\kappa\theta}.\cos\left\{\frac{\Omega_\kappa\theta}{1+\frac{1}{2}\beta_\kappa^{-2}}+\arg(\mathrm{F}_\kappa)\right\}\right]$$

with

$$\mathrm{F}_\kappa\equiv\frac{1-i\{1+\beta_\kappa^{-2}\}^{-\frac{1}{2}}}{1-i\{\beta_\kappa^{-2}+(2\beta_\kappa^2+1)\Delta_\kappa^2\}\{1+\beta_\kappa^{-2}\}^{-1/2}}. \quad \ldots \quad (71)$$

The results (19)–(71) are summarized by the following theorem reciprocal to (18):—

Let $f_1(z)$ and $f_2(z)$ be two image-functions regular at $x>0$ which fulfil the condition

$$|r_1(\omega)-r_2(\omega)|\leqslant\delta_0 \quad \text{for} \quad \omega\leqslant\omega_0.$$

The originals $h_1(t)$, $h_2(t)$ are subjected to the operation

$$\bar{h}(t_0\theta\,;\,t_0)=\frac{q_0}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-q_0^2(\tau-\theta)^2}h(t_0\tau)d\tau,$$

with $q_0=2{\cdot}03_8$ (determined by (31)),

$$t_0=\frac{\omega_0}{2q_0\beta_0}, \quad \theta\equiv\frac{t}{t_0}\gtrsim 1 \quad \text{and} \quad 1\lesssim\beta_0\lesssim 2{\cdot}4 \quad \text{for} \quad \theta\gg 1.$$

Further, let

$$|r_1(\omega)-r_2(\omega)|\leqslant\delta_0\left|\rho\left(\frac{\omega}{\omega_0}-1\right)\right| \quad \text{for} \quad \omega>\omega_0$$

with

$$\rho(x)=p(1+\rho_1x+\rho_2x^2+\ldots)+\sum_{(\kappa)}\frac{a_\kappa d_\kappa}{d_\kappa^2+(x-x_\kappa)^2}$$

$$\equiv\rho_{\mathrm{P}}(x)+\sum_{(\kappa)}\rho_\kappa(x)\,;\quad \rho(0)=1,\ \rho_1>0,$$

$$\left[\left|\frac{d_\kappa}{a_\kappa}\rho(x_\kappa)\right|-1\right]^2\lesssim 10^{-1}$$

$$\text{and} \quad \beta_0^2\Omega_\kappa d_\kappa<10^{1/4}\left|\frac{d_\kappa}{a_\kappa}\rho(x_\kappa)-1\right|;\ \Omega_\kappa\equiv\frac{\omega_\kappa}{\omega_0}.$$

Then the inequality holds

$$\frac{1}{\delta_0}\,h_1(t_0\theta\,;\,t_0)-\bar{h}_2(t_0\theta\,;\,t_0)<\Phi\left(\frac{\theta}{2\beta_0}\right)-\epsilon_{P^{(0)}}+p\;\epsilon_{P^1}+\;\epsilon_\kappa$$

with

$$\epsilon_P=\frac{2}{\pi}\,e^{-\beta_0\theta}\sum_{\nu=1}^{\nu}\frac{A_\nu\,.\,e_\nu(B)\,.\sin\,(\theta+\nu\phi)}{\nu+1},$$

$$Be^{i\phi}\equiv(2\beta_0{}^2+1-\rho_1)+i\theta,\quad e_n(x)=\frac{n!}{x^n}\left\{1-e^{-x}\sum_{\nu=0}^{n-1}\frac{x^\nu}{\nu!}\right\},$$

$$A_1=1,\quad A_2=0,\quad A_3=-(\beta_0{}^2-\tfrac{1}{2})-(\tfrac{1}{2}\rho_1{}^2-\rho_2),$$

$$A_4=-\tfrac{1}{8}-(\tfrac{1}{8}\rho_1{}^3+\rho_1\rho_2-\rho_3),\quad A_5=\tfrac{1}{2}(\beta_0{}^2-\tfrac{1}{2})^2+(\tfrac{1}{2}\rho_1{}^2-\rho_2)\beta_0{}^2+(-\tfrac{1}{8}\rho_1{}^4+\tfrac{1}{2}\rho_1{}^2\rho_2+\tfrac{1}{4}\rho_2{}^2-\rho_1\rho_3+\rho_4)\;\ldots,$$

and the affix $^{(0)}$ indicating $\rho_1=\rho_2=\ldots=0$; further,

$$\epsilon_\kappa=\sum_{(\kappa)}\frac{2a_\kappa}{\Omega_\kappa}\left\{e^{-\Delta_\kappa\Omega_\kappa\theta+(\beta_\kappa\Delta_\kappa)^2}\,.\sin\,[\Omega_\kappa\theta-(2\beta_\kappa{}^2+1)\Delta_\kappa]-\Delta_\kappa|\,F_\kappa|\,e^{-\frac{\sqrt{1+\beta_\kappa{}^{-2}}}{2\beta_\kappa{}^2+1}\Omega_\kappa\theta}\,.\cos\left[\frac{\Omega_\kappa\theta}{1+\frac{1}{2}\beta_\kappa{}^{-2}}+\arg\,(F_\kappa)\right]\right\},$$

with

$$\beta_\kappa\equiv\beta_0\Omega_\kappa,\quad \Delta_\kappa\equiv\frac{d_\kappa}{\Omega_\kappa},\quad\text{and}$$

$$F_\kappa\equiv\frac{1-i\{1+\beta_\kappa{}^{-2}\}^{-1/2}}{1-i\{\beta_\kappa{}^{-2}+(2\beta_\kappa{}^2+1)\Delta_\kappa{}^2\}\{1+\beta_\kappa{}^{-2}\}^{-1/2}}\,.\qquad\ldots\quad(72)$$

There is an analogue to the theorem (17 *a*), (18 *a*) concerning driving-point functions only. The corresponding information

$$\left|1-\frac{r_2(\omega)}{r_1(\omega)}\right|\leqslant\Delta_0\text{ for }\omega\leqslant\omega_0\quad\ldots\quad(19\,a)$$

yields the inequality

$$\left|\frac{h_1(t_0\theta\,;\,t_0)-\bar{h}_2(t_0\theta\,|\,;\,t_0)}{\bar{h}_1(t_0\theta\,;\,t_0)}\right|<\Delta_0+\frac{2}{\pi\bar{h}_1(t\theta_0\,;\,t_0)}\times\left\{\int_1^\infty\{r_1(\omega_0\Omega-r_2(\omega_0\Omega)\}e^{-\left(\frac{a_0}{2q_0}\right)^2\Omega^2}\frac{\sin\theta\Omega}{\Omega}\,d\Omega\right|-\Delta_0\int_1^\infty r_1(\omega_0\Omega)e^{-\left(\frac{a_0}{2q_0}\right)^2\Omega^2}\frac{\sin\theta\Omega}{\Omega}\,d\Omega\Bigg\},\qquad\ldots\quad(48\,c)$$

whose evaluation follows the same line as (48 *b*).

The theorems (18) and (72) combined give relations between the indicial admittances $h_{1,2}(t)$ belonging to two transfer functions $f_1(z)$ and $f_2(z)$ whose ohmic parts differ by less than δ_0 in a finite frequency interval. As pointed out in the introduction, corresponding relations between more general transients in the two systems are wanted which are excited by equal signal inputs. Although by (22) any transient can be found when $h(t)$ and the exciting signal are given, the theorem (72) does not apply to the case of a general excitation, as it does not concern $h(t)$, but mean value functions of $h(t)$.

This difficulty can be eliminated if the information about $f_1(z)$ and $f_2(z)$ in the main frequency range is of a more general kind.

Let us, *e. g.*, suppose that $f_1(z)$ belongs to a given physical system, while $f_2(z)$ is an auxiliary function (both fulfilling (2), (3), and (8 *d*)) substituted for facilitating the calculation of the following transient problem. A certain (unknown) excitation of $S_I(t)$ causes a response $S_{II}(t)$ which will be similar in the actual and the auxiliary system, and error limits corresponding to the theorems (18) and (72) are to be established between them. The response $S_{II}(t)$ is given in the system 2, say. Let $g_{II}(z)$ be the signal spectrum corresponding to $S_{II}(t)$, *i. e.*,

$$g_{II}(z) \doteqdot S_{II}(t), \quad . \quad . \quad . \quad . \quad . \quad (73\,a)$$

and likewise

$$g_I(z) \doteqdot S_I(t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (73\,b)$$

We assume

$$S_1(t)=0 \text{ for } t<0, \text{ i. e., } g_1^{(z)} \text{ regular for } z>0\,; \quad . \quad (73\,c)$$

then the same holds for S_{II}, g_{II}. Both systems being excited by the same signal $S_I(t)$; $g_I(z)$, (21) gives the image functions belonging to the two systems:

$$\left.\begin{aligned} g_2(z)&=g_{II}(z) \qquad \text{for the system II.,}\\ g_1(z)&=\frac{f_1(z)}{f_2(z)}\,g_{II}(z) \;\;,,\;\; ,, \;\; ,, \;\; \text{I.}\end{aligned}\right\} \quad . \quad . \quad (74)$$

Therefore, $g_{II}(z)$ being given, an information of the form

$$f_2(z) \doteqdot f_1(z)\{1+\delta(z)\}$$

with

$$|\,\delta(z)\,| \leqslant \delta_0^* \quad \text{for} \quad z=i\omega, \quad \omega_1 \leqslant \omega \leqslant \omega_2, \text{ say,}$$

will yield a corresponding relation between

$$R_e(g_2(i\omega)) \quad \text{and} \quad R\,(g_1(i\omega)).$$

If this is brought into a form of the kind (17) or (19) respectively, relations of the kind (18) or (72) respectively arise concerning the more general transients

$$S_{II}^{(1)}(t) \div g_1(z) \text{ and } S_{II}^{(2)}(t) = S_{II}(t) \div g_2(z) \quad . \quad . \quad (74\,a)$$

instead of $h_1(t)$ and $h_2(t)$.

Of special interest are transients due to excitations with a stationary sinusoidal state of the frequency ω_0, *i. e.*, with one imaginary pole pair $\pm i\omega_0$ in their signal spectrum in lieu of the zero pole of the unit jump. When applied to two systems whose transfer functions $f_1(z)$, $f_2(z)$ are equal for $z = \pm i\omega_0$ and similar near $\pm i\omega_0$, they will excite transients which become similar after a certain time has elapsed, and the stationary frequency becomes noticeable. This time will depend on $f_1(i\omega)$, $f_2(i\omega)$ and on the frequency range about ω_0 where the relative difference is small. When extending the above calculations to this case, it is to be expected that the mean value operation (20 *a*) must be replaced by a more general process which permits the frequency ω_0 to predominate instead of $\omega = 0$. It will be of special use for the calculation of transients errors in selective systems when employing the substitution of " ideal filters " as a first step of approximation [23].

The following discussion concerning excitations with stationary sinusoidal state will be kept on general lines, omitting detailed calculations (to be carried out similarly to those leading to (72)). The simple excitations $\cos \omega_0 t$ and $\sin \omega_0 t$ for $t \geqslant 0$ are considered only, more general signal envelopes are obtained by multiplying the signal spectrum with a corresponding term. Borel's theorem (21) gives at once Fourier integrals for the transients $h_{\cos}$ and $h_{\sin}$ belonging to ω_0 ;

$$\left.\begin{aligned} h_{\cos}(\theta) &= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\zeta}{\zeta^2+1} f(\omega_0\zeta) e^{\theta\zeta}\, d\zeta, \\ h_{\sin}(\theta) &= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{f(\omega_0\zeta)}{\zeta^2+1} e^{\theta\zeta}\, d\zeta \end{aligned}\right\} . \quad (75\,a, b)$$

with $\qquad \theta \equiv \omega_0 t, \quad \zeta \equiv \dfrac{z}{\omega_0}$.

[23] See [12], and [14], part 3, para. 5. Further, H. Salinger, *E. ektr. Nachr. Techn.* vi. p. 293 (1929).

Similarly as below (10) the real integral forms

$$\left.\begin{aligned} h_{\cos}(\theta) &= \frac{2}{\pi}\int_0^\infty u(i\omega_0\eta)\,\frac{\eta\sin\theta\eta - \sin\theta}{\eta^2-1}\,d\eta, \\ h_{\sin}(\theta) &= \frac{2}{\pi}\int_0^\infty u(i\omega_0\eta)\,\frac{\cos\theta - \cos\theta\eta}{\eta^2-1}\,d\eta \end{aligned}\right\}. \quad (76\,a, b)$$

are obtained ; they have to be written as Stieltjes integrals in analogy to (15) when including imaginary poles of $f(z)$. For the sake of brevity this will be omitted. When deriving (76), use is made of the Poisson integral

$$v(i\omega_0) = \frac{2}{\pi}\int_0^\infty \frac{u(i\omega_0\eta)}{\eta^2-1}\,d\eta, \quad . \quad . \quad . \quad . \quad (77)$$

which follows at once from (14) on the understanding that the residue at $\eta=1$ is not taken into account, *e. g.*, as the limit

$$v(i\omega_0) = \frac{2}{\pi}\lim_{T\to\infty}\int_0^\infty u(i\omega_0\eta)\,\frac{1-\cos T(\eta-1)}{\eta^2-1}\,d\eta. \quad (77\,a)$$

The integrand of both (76 *a*) and (76 *b*) is finite for $\eta=1$.

(76) must yield the stationary state for $\theta\to\infty$. This is verified at once, *e. g.*, for (76 *a*). The part due to the term $\eta\sin\theta\eta$ becomes negligible for $\theta\to\infty$, with the exception of a small interval (of the order $\frac{1}{\sqrt{\theta}}$, say) about $\eta=1$. Writing $\eta=1+\epsilon$, we have

$$\begin{aligned} \lim_{\theta\to\infty} h_{\cos}(\theta) = \frac{2}{\pi}\lim_{\theta\to\infty}\Big[\sin\theta\Big\{ &-\int_0^\infty u(i\omega_0\eta)\,\frac{1-\cos\theta(\eta-1)}{\eta^2-1}\,d\eta \\ &+\frac{1}{2}\int_{-1}^\infty u(i\omega_0(1+\epsilon))\,\frac{\cos\theta\epsilon}{1+\epsilon/2}\,d\epsilon\Big\} \\ &+\frac{1}{2}\cos\theta\int_{-1}^\infty u(i\omega_0(1+\epsilon)\,\frac{1+\epsilon}{1+\epsilon/2}\,\frac{\sin\theta\epsilon}{\epsilon}\,d\epsilon\Big] \\ = -v(i\omega_0).\sin\theta + 0 + \frac{1}{\pi}u(i\omega_0)\cos\theta &\int_{-\infty}^{+\infty}\frac{\sin u}{u}\,du \\ &\text{(by virtue of (77 } a\text{)} \\ = u(i\omega_0).\cos\theta - v(i\omega_0).\sin\theta. & \end{aligned}$$

A similar calculation yields asymptotic approximations for large θ, *i. e.*, the deviations from the stationary state. When dealing with two functions $f_1(\zeta)$ and $f_2(\zeta)$ which

are equal at $\zeta=i$ ($\eta=1$), the difference $(h^{(1)}_{\substack{\cos\\\sin}}-h^{(2)}_{\substack{\cos\\\sin}}$ has no stationary term [24], and the corresponding expansion tends towards 0 for $\theta\to\infty$. Writing, in analogy to (54),

$$\rho(\eta)\equiv u_1(i\omega_0\eta)-u_2(i\omega_0\eta) \quad . \quad . \quad . \quad (78\,a)$$

and

$$h^{(1)}_{\substack{\cos\\\sin}}(\theta)-h^{(2)}_{\substack{\cos\\\sin}}(\theta)\equiv \Delta h_{\substack{\cos\\\sin}}(\theta),$$

we have

$\rho(1)=0$. *i. e.*, $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\rho(\eta)}{\eta^2-1}\,d\eta=0$, by virtue of (77) ;

. . . (79 *a*, *b*)

this yields the equations

$$\left.\begin{aligned}\Delta h_{\cos}(\theta)&=\frac{2}{\pi}\int_0^\infty \rho(\eta)\sin\theta\eta\,\frac{\eta d\eta}{\eta^2-1},\\ \Delta h_{\sin}(\theta)&=-\frac{2}{\pi}\int_0^\infty \rho(\eta)\cos\theta\eta\,\frac{d\eta}{\eta^2-1},\end{aligned}\right\} \quad . \quad . \quad (80\,a, b)$$

It will depend on the information supplied and the character of $\rho(\eta)$ how easily satisfactory approximations of these integrals can be obtained. A division of the frequency interval into three ranges suggests itself : a " main " range about $\eta=1$ and the adjacent intervals of low and high frequencies. In the main interval $(1-\epsilon_1)\leqslant\eta\leqslant(1+\epsilon_2)$, $\rho(\eta)$ will be given by a power expansion

$$\rho(1+\epsilon)=\rho_1\epsilon+\rho_2\epsilon^2+\ldots;\ \eta=1+\epsilon,\ -\epsilon_1\leqslant\epsilon\leqslant\epsilon_2,$$
by virtue of (79 *a*), (81 *a*)

thus yielding the contributions

$$\left.\begin{aligned}&\frac{1}{\pi}\int_{-\epsilon_1}^{+\epsilon_2}\{\rho_1+(\rho_2+\tfrac{1}{2}\rho_1)\epsilon+(\rho_3+\tfrac{1}{2}\rho_2-\tfrac{1}{4}\rho_1)\epsilon^2\ldots\}\\ &\qquad\qquad\times\sin\theta(1+\epsilon)\,d\epsilon,\\ -&\frac{1}{\pi}\int_{-\epsilon_1}^{+\epsilon_2}\{\rho_1+(\rho_2-\tfrac{1}{2}\rho_1)\epsilon+(\rho_3-\tfrac{1}{2}\rho_2+\tfrac{1}{4}\rho_1)\epsilon^2\ldots\}\\ &\qquad\qquad\times\cos\theta(1+\epsilon)\,d\epsilon\end{aligned}\right\}$$
. . . (82 *a*, *b*)

[24] A small difference δ_0 which would correspond to the conditions of the theorems (18) and (72) may be introduced in the following, but is dropped for the sake of simplicity.

to (80), which are elementary integrals. Considering an interval $0 \leqslant \eta \leqslant c\theta^{-1}$ with c^2 of the order 10, say, the power expansions of the harmonic terms may be used in (80). If, *e. g.*, a power expansion of ρ in η exists having a corresponding range of convergence, this leads to convergent expansions in θ^{-1}, starting with the second order term in (80 *a*), with the first order term in (80 *b*). On the other hand, for the interval $\eta \gtrsim c\theta^{-1}$ semiconvergent expansions in θ^{-1} can be obtained by continued partial integration with regard to the harmonic terms. The sufficiency of such expansions can be decided in an actual case only. Conditions will vary with the character of $\rho(\eta)$ and the time θ. For late times, $\theta >> 1$, approximations will always be obtainable in a simple way; for early times, $\theta << 1$, on the other hand, the behaviour of $\rho(\eta)$ at high frequencies matters only, an expansion of ρ in η^{-1} yielding the associated expansion for $\Delta h_{\substack{\cos \\ \sin}}$ in θ. Generally, more information about $\rho(\eta)$ is required at high frequencies for cos-excitation, while the opposite is true for sin-excitation.

If the main range does not give the bulk contribution [(82)], and the evaluation of (80) or (76) respectively offers difficulties arising from the character of $\rho(\eta)$ or from lack of information about $u(i\omega_0\eta)$ outside the main range, the problem is the same—*mutatis mutandis*—as it was for the transients $h(t)$. Excessive contribution of the secondary ranges has been prevented in (18) by limiting the time $(\omega_0 t)$ in (72) by applying the mean value process (20 *a*). Dealing correspondingly with the present problem means stressing an intermediate frequency range, *i. e.*, multiplying the integrand of (75) by a term $m(\zeta)$ having a pair of finite saddle-points. According to (20 *d*), (24) multiplication means applying the process (20 *a*) to $h_{\substack{\cos \\ \sin}}(\theta)$ with the momentum function

$$\mu(\theta) \doteqdot \omega_0 \zeta . m(\zeta). \quad \text{(Fourier transform of } m(\zeta). \quad . \quad (23\,a)$$

As $\mu(\theta)$ must be even,

$$\mu(-\theta) = \mu(\theta), \quad . \quad . \quad . \quad . \quad . \quad (20\,\beta)$$

it is

$$Im[m(i\eta)] = 0, \quad \text{i. e., } m(-i\eta) = m(i\eta), \quad . \quad . \quad (83)$$

and the operation remains multiplicative when going over to the real Fourier integral (see (32) and (40)),

$$\Delta\bar{h}_{\cos}(\theta)=\frac{2}{\pi}\int_0^\infty m(i\eta)\rho(\eta)\sin\theta\eta\,\frac{\eta\,d\eta}{\eta^2-1},$$
$$\Delta\bar{h}_{\sin}(\theta)=-\frac{2}{\pi}\int_0^\infty m(i\eta)\rho(\eta)\cos\theta\eta\,\frac{d\eta}{\eta^2-1}. \quad . \quad . \quad (84)$$

So far the process is the same as above. There is, however, the important difference that the second and third equation of (20, *b*) are no longer valid. For supposing that $m(i\eta)$ has a maximum at $\eta=1$, say, its Fourier transform necessarily oscillates; likewise the condition $m(0)=1$ (41) does not hold any longer, since

$$m(i)=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

i. e., invariance of the stationary sinusoidal state ω_0 under the operation (20 *a*) is required, and, in general, $m(0)\neq m(i)$. (In any case, the third condition of (20 *b*) is meaningless without the second one.)

This change of conditions does not impair the physical soundness of the process (20 *a*). It is merely the consequence of the change from the unit to the unit sinusoidal excitation, and offers the same adaptability of the information gained about the transient to the given information supplied about the frequency response, $\bar{h}_{\substack{\cos\\\sin}}(\theta)$ being considered as a degree of information inferior to $h_{\substack{\cos\\\sin}}(\theta)$.

The considerations about the choice of a special momentum function which resulted in (46) apply here likewise, and lead to the adoption of

$$m(\zeta)=\text{const.}\ \tfrac{1}{2}\{e^{\frac{1}{2}\beta^2(\zeta+i\eta^{(0)})^2}+e^{\frac{1}{2}\beta^2(\zeta-i\eta^{(0)})^2}\}.$$

When introducing equivalent constants and accounting for (85) this may be written

$$m(\zeta)=\frac{e^{\frac{1}{2}\beta^2(\zeta^2+1)}\cos\alpha\zeta}{\cosh\alpha};\quad m(i\eta)=\frac{e^{\frac{1}{2}\beta^2(1-\eta^2)}\cosh\alpha\eta}{\cosh\alpha}. \quad . \quad (86)$$

The corresponding momentum function (23 *a*) is

$$m(\theta)=\frac{\omega_0 e^{\frac{1}{2}\left[\beta^2+\left(\frac{\alpha}{\beta}\right)^2\right]}}{\sqrt{2\pi}\beta\cosh\alpha}e^{-\frac{1}{2}\left(\frac{\theta}{\beta}\right)^2}\cos\frac{\alpha}{\beta^2}\theta. \quad . \quad . \quad (87)$$

The equation of the saddle-points of (86) is

$$m'(\zeta)=0\,; \quad \text{or} \quad \left(\frac{\beta}{\alpha}\right)^2=\frac{\operatorname{tg}\alpha\zeta}{\alpha\zeta} \text{ and } \zeta=0. \quad . \quad (88)$$

Its solutions converge for $\alpha\to 0$ and finite β into the essential singularity $\zeta=\infty$ (with the exception of $\zeta=0$), which leads back to (46). For

$$\alpha<\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89\,a)$$

(88) has real solutions ξ only, viz., alternative maxima and minima when proceeding along the real axis; they do not concern our problem directly. $m(i\eta)$ has its maximum still at $\eta=0$, but decreases more slowly than for $\alpha=0$. For

$$\alpha=\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89\,b)$$

the maximum at $\eta=0$ is of fourth order corresponding to a confluent saddle-point at $\zeta=0$, with steepest descents under $0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi$. For

$$\alpha>\beta_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89\,c)$$

there is one pair of saddle-points $\pm i\eta_0$ at the imaginary axis causing a maximum of $m(i\eta)$. Its equation is

$$\left(\frac{\beta}{\alpha}\right)^2=\frac{\operatorname{tgh}\alpha\eta_0}{\alpha\eta_0}(\leqslant 1). \quad . \quad . \quad . \quad . \quad (88\,a)$$

If η_0 equals the stationary frequency, it follows

$$\beta=\sqrt{\alpha.\operatorname{tgh}\alpha}\,; \; \beta=\alpha\left(1-\frac{\alpha^2}{6}+\frac{19}{360}\alpha^4\ldots\right)=\sqrt{\alpha}(1-e^{-2\alpha}\ldots). \quad . \quad . \quad . \quad (90)$$

α and β will be replaced by two equivalent constants of direct physical meaning, viz., η_0 and a parameter measuring the oscillation of the momentum function. Recalling the value of q_0 (31), (46) which relates the Gaussian function to the basic function (29), we denote by ν the number of periods of the cos-term of (87) in the interval

$$-\frac{q_0}{\sqrt{2}}\leqslant\frac{\theta}{\beta}\leqslant+\frac{q_0}{\sqrt{2}}$$

of the rectangular function, which is equivalent to the exponential term. It follows

$$\nu=\frac{g_0}{\pi\sqrt{2}}\cdot\frac{\alpha}{\beta}=0{\cdot}459\,\frac{\alpha}{\beta}. \quad . \quad . \quad . \quad . \quad . \quad (91)$$

It is seen that the " non-oscillatory " range

$$\nu\leqslant 0{\cdot}5 \quad . \quad . \quad . \quad . \quad . \quad . \quad (92\,a)$$

coincides very nearly with the range

$$\nu\leqslant 0{\cdot}459 \quad . \quad . \quad . \quad . \quad . \quad . \quad (92\,b)$$

of monotony of $m(i\eta)$. Defining (89 a),

$$\left.\begin{aligned} &g(y)=x(y)=\text{inverse function to } y=\frac{\operatorname{tgh} x}{x}\leqslant 1\,;\\ &g(y)=\sqrt{3(1-y)}\{1+\tfrac{3}{5}(1-y)\ldots\} \text{ for } y\sim 1,\\ &g(y)\approx\frac{\operatorname{tgh}(y^{-1})}{y} \text{ for } y\lesssim 0{\cdot}6, \end{aligned}\right\} \quad (93)$$

we have

$$\alpha=\frac{1}{\eta_0}\cdot g\left(\left(\frac{0{\cdot}459}{\nu}\right)^2\right),\quad \beta=\frac{0{\cdot}459}{\nu\eta_0}\cdot g\left(\left(\frac{0{\cdot}459}{\nu}\right)^2\right). \quad (94)$$

For $\nu\gtrsim 0{\cdot}6$ it is approximately

$$\alpha\approx 4{\cdot}75\,\frac{\nu^2}{\eta_0}\operatorname{tgh}\left(\left(\frac{\nu}{0{\cdot}459}\right)^2\right),\quad \beta\approx 2{\cdot}18\,\frac{\nu}{\eta_0}\operatorname{tgh}\left(\left(\frac{\nu}{0{\cdot}459}\right)^2\right). \quad . \quad . \quad . \quad (94\,a)$$

Since the main frequency range is supposed to include the stationary frequency, it will usually be

$$\eta_0\sim 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

The optimum choice of η_0 will depend on the relative importance of the upper and lower frequency ranges and the time θ, but not on ν if $\nu\gtrsim\frac{3}{4}$, because then $m(i\eta)$ is nearly symmetrical about η_0 in an interval where

$$\frac{m(i\eta)}{m(i\eta_0)}\lessapprox\gtrapprox 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

Thus the parameters η_0 and ν may (usually) be fixed

independently, which is not true for α and β. θ is restricted by the condition

$$\theta \gtrsim \frac{2\pi}{\eta_0}\nu, \qquad \ldots\ldots\ldots \quad (97)$$

equivalent to (51 *a*), which prevents obtaining "empty" statements when applying (86), (87) [25].

The result of applying (86) to (84) is

$$\Delta h_{\cos}(\theta;\alpha,\beta) = \frac{2}{\pi}\frac{e^{\frac{1}{2}\beta^2}}{\cosh\alpha}\int_0^\infty \rho(\eta)e^{-\frac{1}{2}\beta^2\eta^2}\cosh\alpha\eta\sin\theta\eta\,\frac{\eta\,d\eta}{\eta^2-1},$$

$$\Delta\bar{h}_{\sin}(\theta;\alpha,\beta) = -\frac{2}{\pi}\frac{e^{\frac{1}{2}\beta^2}}{\cosh\alpha}\int_0^\infty \rho(\eta)e^{-\frac{1}{2}\beta^2\eta^2}\cosh\alpha\eta\cos\theta\eta\,\frac{dy}{\eta^2-1}; \qquad \ldots \quad (98\,a,\,b)$$

or in terms of ν, η_0 when supposing $\nu \gtrsim 0{\cdot}6$, and putting

$$\nu^* \equiv \nu.\mathrm{tgh}\left(\left(\frac{\nu}{0{\cdot}459}\right)^2\right) \approx \nu \text{ for } \nu \gtrsim \tfrac{3}{4}, \qquad . \quad (99\,c)$$

$$\Delta\bar{h}_{\cos}(\theta;\eta_0,\nu) = \frac{2}{\pi}\frac{e^{4\cdot75\cdot\frac{1}{2}\left(\frac{\nu^*}{\eta_0}\right)^2}}{\cosh\left(4{\cdot}75\dfrac{\nu\nu^*}{\eta_0}\right)}$$

$$\times\int_0^\infty \rho(\eta)e^{-4\cdot75\cdot\frac{1}{2}\nu^{*2}\left(\frac{\eta}{\eta_0}\right)^2}\cosh\left(4{\cdot}75\nu\nu^*\frac{\eta}{\eta_0}\right)\sin\theta\eta\,\frac{\eta\,d\eta}{\eta^2-1},$$

$$\Delta\bar{h}_{\sin}(\theta;\eta_0,\nu) = -\frac{2}{\pi}\frac{e^{4\cdot75\cdot\frac{1}{2}\left(\frac{\nu^*}{\eta_0}\right)^2}}{\cosh\left(4{\cdot}75\dfrac{\nu\nu^*}{\eta_0}\right)}$$

$$\times\int_0^\infty \rho(\eta)e^{-4\cdot75\cdot\frac{1}{2}\nu^{*2}\left(\frac{\eta}{\eta_0}\right)^2}\cosh\left(4{\cdot}75\nu\nu^*\frac{\eta}{\eta_0}\right)\cos\theta\eta\,\frac{d\eta}{\eta^2-1}. \qquad \ldots \quad (99)$$

The methods by which these integrals may be evaluated have been discussed above; therefore there is no need for details here. To give, however, a simple example, let us suppose that $\rho(1+\epsilon)$ is of relatively small variation, *i. e.*, that (81) is valid in a range (96), beyond which contribution to (99) may be neglected. We assume further

[25] For small values of θ this process is not required, satisfactory results being obtainable by power expansion, as pointed out above, (51 *c*), (66 *a*), and (82).

$\nu \gtrsim \frac{3}{4}$, *i. e.*, $\nu^* = \nu$ and $\eta_0 = 1$. Then ordinary saddle-point integration yields the first approximations :

$$\left.\begin{aligned} \Delta h_{\cos}(\theta;\,1,\nu) &\approx \rho_1 \frac{e^{-\frac{\theta^2 - a_2\theta^2}{9{\cdot}5\nu^2 - 4a_3}}}{\sqrt{\pi\left(4{\cdot}75\,\frac{\nu^2}{2} - a_3\right)}} \\ &\qquad \times \sin \frac{2a_2\theta}{9{\cdot}5\nu^2 - 4a_3}, \\ \Delta \bar{h}_{\sin}(\theta;\,1,\nu) &\approx -\rho_1 \frac{e^{-\frac{\theta^2 - b_2\theta^2}{9{\cdot}5\nu^2 - 4b_3}}}{\sqrt{\pi\left(4{\cdot}75\,\frac{\nu^2}{2} - b_3\right)}} \\ &\qquad \times \cos \frac{2b_2\theta}{9{\cdot}5\nu^2 - 4b_3}, \end{aligned}\right\} \quad \theta \gtrsim 2\pi\nu$$

with

$$a_2 \equiv \frac{\rho_2}{\rho_1} + \frac{1}{2}, \qquad b_2 \equiv \frac{\rho_2}{\rho_1} - \frac{1}{2},$$

$$a_3 \equiv \frac{\rho_3}{\rho_1} - \frac{1}{2}\left(\frac{\rho^2}{\rho_1}\right)^2 - \frac{3}{8}, \quad b_3 \equiv \frac{\rho_3}{\rho_1} - \frac{1}{2}\left(\frac{\rho_2}{\rho_1}\right)^2 + \frac{1}{8}. \qquad \ldots \quad (100)$$

They represent rapidly damped oscillations. When considering higher order terms of (81) and accounting for (99 *c*), (100) is continued as semi-convergent expansion with oscillating terms. Further correction is obtained when accounting for the limited range of convergence of (81) ; then the exponential functions of square argument are replaced by Gram-Charlier compounds, which are analogous to the functions (64). If components of large fluctuation are present in $\rho(\eta)$, the process of steepest descent is adapted correspondingly. As shown above (see (68) etc.), it results in additional oscillatory terms with linear attenuation exponents.

All the results obtained about the errors of transient phenomena are based on relations between the ohmic parts of the associated frequency characteristics. As mentioned above (after (12)), similar results may be derived in terms of the reactive parts. For many cases, however, it would be desirable to draw corresponding conclusions from the absolute values of frequency characteristics, since it is usually easier to obtain them experimentally than either of the Cartesian components.

Unfortunately, however, these possibilities are very limited, for the knowledge of $|f(z)|$ at the imaginary axis is, in general, not sufficient for that of the real parts, if merely (2) and (3)—even for $x=0-$ are fulfilled. This is physically evident. *E. g.*, it is well known that two communication systems may have equal gain curves but different phase-characteristics. Relations of the desired kind exist for driving-point functions $fd(z)$ only for the conditions (7) and (5) ensure that no zero exists for $x>0$. Supposing, for the sake of simplicity, that the equality sign does not hold in (5) (*i. e.*, that dissipation is present; the general case requires a limit process), this leads to the Poisson integral,

$$\left.\begin{matrix} u_d(iy) \\ v_d(iy) \end{matrix}\right\} = |f_d(iy)| \cdot \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} \left[\frac{2}{\pi} y \int_0^\infty \frac{\log |f_d(iy)|}{\eta^2 - y^2} d\eta\right], \quad . \quad . \quad . \quad (101)$$

with the same understanding about the residue at $\eta=y$ as in (77), (77 *a*).

LXXI. *Absorption of Soft X-rays in Aluminium.* *By* G. B. Bandopadhyaya, *M.Sc., Ph.D., Assistant Professor of Physics, Science College, Patna (India), and* A. T. Maitra, *M.Sc., Research Scholar, Government of Bihar and Orissa (India)* *.

[Plates XV. & XVI.]

Abstract.

1. With a grating spectrometer used at grazing incidence, soft X-ray lines, due to a number of anticathodes, have been photographed from 7 Å. to 25 Å.

2. The mass absorption coefficients have been determined accurately for different wave-lengths with a recording microphotometer.

3. The experimental results go to show that Jönsson's Universal Absorption Curve and Stobbe's Absorption Laws based on Quantum Mechanics do not represent the present data quite satisfactorily.

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* Communicated by the Authors.

I. *Introduction.*

MEASUREMENTS of mass absorption coefficients with homogeneous X-rays were made by Bragg and Pierce [1] for the first time. The theoretical importance attached to the phenomenon of X-ray absorption has attracted a large number of experimenters to determine the mass absorption coefficients of X-rays in a large number of absorbing substances. Sufficient work in this connexion has not, however, been done in the soft X-ray region. Holweck [2] has determined absorption in a number of foils and gases for heterogeneous beams of X-rays with voltages of the order of 100. One of us (G. B. B.)[3], in course of an investigation by a photoelectric method, found that the intensity of total soft X-rays at 350 volts falls to 0·6 per cent. by transmission through $1{\cdot}36\times10^{-5}$ cm. of gold and to 0·1 per cent. by transmission through $7{\cdot}9\times10^{-5}$ cm. of aluminium. Jönsson[4], using a photographic method, has measured the absorption coefficients for X-rays up to 12 Å. in aluminium, copper, silver, and platinum. Dreshem and Shein [5] have evaluated the absorption coefficients of different gases for carbon K_α-line at 44 Å. Woernle [6] worked with gases from 2 Å. to 12 Å. The present work was taken up with a view to determine the absorption of soft X-rays from 7 Å. to 25 Å. in aluminium by a photographic method.

II. *Description of Experimental Arrangements.*

The grating spectrometer employed in this investigation was of the type devised by Siegbahn and Magnusson [7], and was constructed by Adam Hilger. X-rays issuing from the anticathode of an electron tube of Siegbahn type were limited by two fine slits 0·05 mm. in width and fell on a plane glass grating with a steel wedge in front, the distance of separation between the grating and the wedge was adjusted to be 0·04 mm. The direct, the reflected, and the diffracted rays were recorded over a photographic plate at a distance of 255 mm. from the steel wedge. The plane grating used was ruled at Upsala, having 900 lines in a space 3 mm. wide.

Small leaks over the body of the spectrometer delayed the investigation considerably. It was finally avoided

by warming up the entire body of the apparatus and applying pezein over the surface. Several coatings of shellac varnish were next applied. This stopped all leaks, and with a suitable pumping arrangement a vacuum of a high order could be secured in a few minutes. For the purpose of pumping an improved type of molecular pump constructed by Leybold, backed by a two stage rotatory oil-pump, was used. The limiting vacuum obtainable with the oil-pump alone was of the order of 10^{-4} mm. of mercury, although the rate of pumping at such low pressures is very small. The molecular pump was connected to the apparatus with a short length of tombac tubing having a bore equal to two inches. A preliminary measurement of the vacuum inside the tube and the spectrometer was made with a sensitive Macleod gauge and was found to be too low to be indicated showing a value of pressure less than 10^{-6} mm. of Hg. The Macleod was, however, disconnected during the investigations.

The electron tube was operated with an Italian Transformer at 3 to 4 k.v. with an electron current varying from 80 to 100 milliamperes. The anticathode was earthed while the voltage was applied to the filament, which was heated by a step down transformer insulated for 10 k.v.

A thin leaf of aluminium about $0{\cdot}5\,\mu$ in thickness free from pin-holes was chosen and interposed in front of the photographic plate. This prevented the fogging of the plate due to scattered white light from tungsten filament. An additional leaf of aluminium, the mass of which was accurately determined, was added so as to intercept the X-rays over one-half of the width of the photographic plate. Thus half the plate recorded X-rays which traversed two leaves of aluminium, while the other half recorded X-rays which traversed a single thickness of aluminium.

The plane grating was set so that the incident rays struck it at a grazing angle nearly equal to $1\frac{1}{2}°$. Fig. 1 (Pl. XV.) exhibits a reproduction of some of the plates taken. A part of each of the spectral lines is due to additional absorption in aluminium, the line of demarcation between the two parts may be seen clearly from the shadow of a wire over which aluminium was put.

III. *Sensitiveness of Photographic Plates and Method of Development.*

A considerable amount of preliminary work was done to find out the suitability of different types of photographic plates in the soft X-ray region from 7 A. to 25 A. Drs. F. C. Chalklin and L. P. Chalklin [8] have found that for 64 A. Ilford Process plates are very suitable. Broili and Kiessig [9] have determined the sensibility of different makes of plates for the carbon K_α-line at 44 A., from which it may be concluded that the order of sensitiveness with ordinary light is preserved in the soft X-ray region also. We tested Ilford Process (H.D. 50), Wratten antiscreen (H.D. 430), and Ilford Golden Isozenith plates (H.D. 1400) in our trial experiments and found larger densities with plates which are faster in the optical region. We therefore employed Golden Isozenith plates, which have fine grains and an extended scale of gradation and are capable of giving negatives remarkably free from chemical fog which is so common with faster varieties. The plates employed were 12 cm. long and nearly 3 cm. wide. For development normal solutions of Tabloid Metolquinol developer of Burroughs Wellcome Co. were employed. This developer we find to have excellent keeping qualities in our tropical climate. The plates were developed in a solution contained in a wide bore test tube immersed in a Dewar flask containing water. The temperature of water in the Dewar and also that of the developer was adjusted at 18° C. and development was continued for exactly five minutes, the plate being continuously turned within the developer. The temperature change during the development amounted to less than 0·1° C. Precisely similar process was resorted to for all the plates.

IV. *Method for Photographic Density Measurements.*

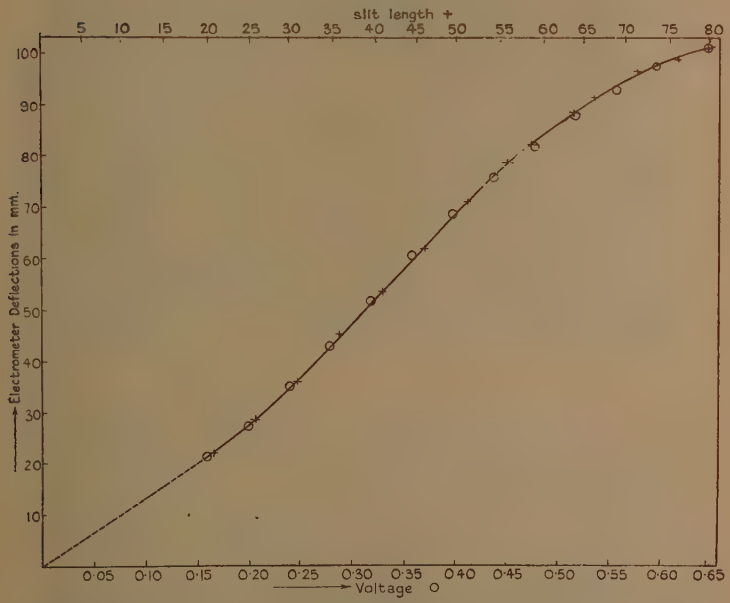
For the measurements of densities of photographic lines a Zeiss recording microphotometer was used. In this apparatus a potassium photoelectric cell and a string electrometer are employed.

Fig. 2 (Pl. XVI.) represents two microphotometer records of nickel L_α-line in the first and second orders. The two peaks to the left side are due to waves suffering additional absorption. The electrometer deflexions are not, however, strictly proportional to string voltages. Hence it is

desirable to calibrate the electrometer at the time of making a record.

Fig. 3 exhibits a typical calibration curve in one of our records. Points marked with circles represent electrometer deflexions as photographed on the recording plate by applying a series of accurately measured potentials to the electrometer fibre, while points marked with crosses were obtained by varying the amount of

Fig. 3.



light falling on the photoelectric cell with a well-calibrated slit with which the apparatus is supplied. Identical results were obtained for the calibration curve by both the methods, as will be seen from fig. 3.

The second method of calibration was generally adopted for the determination of densities.

If I_0 be the amount of light incident on a photographic line, and if I be the amount of light transmitted, then the photographic density D of the line is given by the equation

$$D = \log_{10} \frac{I_0}{I}.$$

For determining the density of a line the electrometer is adjusted to such a sensitivity that an excursion of suitable magnitude may be produced. Then a microphotometer record is made of the line. Next, an exposed part of the negative is taken and a series of electrometer deflexions are also photographed by cutting down the light transmitted to the photoelectric cell to different known fractions. Thus if "r" represents the fraction by which the incident light is to be cut to produce an electrometer excursion equal to that of the peak corresponding to the microphotographic print of a line, then the density of the line can be expressed as $\log_{10}\frac{I}{r}$. This observed density has to be slightly diminished because of the blackening due to white radiation. For this purpose, the densities on the two sides of the line are also measured in a similar manner and the mean value is subtracted from the apparent density. This gives the density produced due to the wave-length corresponding to the line. For each wave-length densities were measured for several different orders of diffraction (usually first three) both for the absorbed and for the unabsorbed portions.

V. *Computation of Intensities.*

In order to determine the relative intensities from density measurements an elaborate method was resorted to. During the main exposure for diffracted rays, which lasted for a number of hours, one half the length of the photographic plate was protected by a thick shutter. Next, the grating was turned round an angle of 180° nearly so as to reflect the incident X-rays towards the side which was protected during the main exposure, and the shutter was turned off. By turning a ground cone a slot covered with thin leaves of aluminium and having a width equal to one-third the breadth of the photographic plate could be brought before the plate. The reflected X-rays were photographed for three different times of exposure at three different parts of the plate after setting the grating at a grazing angle of incidence nearly equal to 1°. This process of giving time exposures for reflected soft X-rays was repeated for three or four settings of the grating, the grating being turned by

some 5′ at a time. In each setting of the grating an exposure for a particular time (usually 40 secs.) was made, while in the two other available positions the duration was varied from a few seconds to a few minutes in multiples of 2. During these exposures current and the voltage in the X-ray tube were adjusted at the same value within $\frac{1}{4}$ per cent.

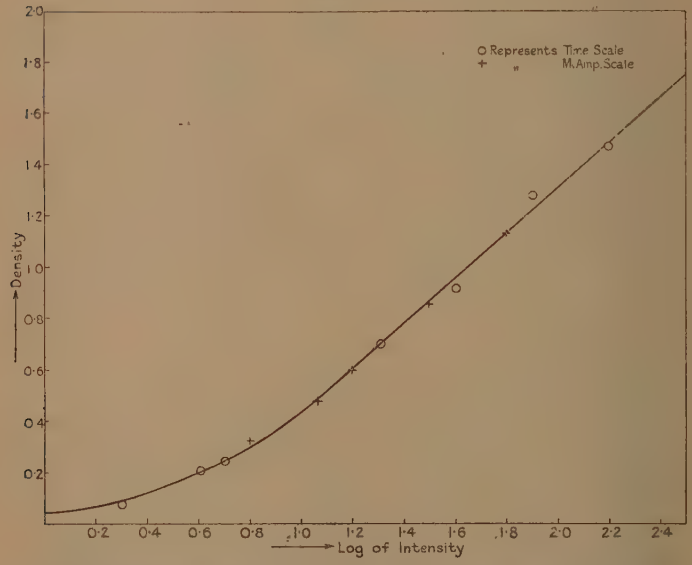
In fig. 4 (Pl. XVI.) the blackening due to reflected rays for different times of exposure may be seen at the left side of the figure. With a grazing angle of incidence larger than 1° rays shorter than 7 Å. are not appreciably reflected from glass, while rays larger than 25 Å. are considerably absorbed in aluminium leaves. Thus the density produced is due to rays from 7 Å. to 25 Å., which was the working range in this investigation. The equal time exposures at different angles gave the small change in intensity of reflexion with variation in the angle of incidence. Assuming density to be proportional to the time of exposure, one can, by measuring the densities of the reflected rays, plot a curve of density against logarithm of intensity. Such a curve was drawn for each of the plates, and this was utilized for computing relative intensities of absorbed and unabsorbed monochromatic radiations.

The intensity is, however, proportional[10] to t^p in the optical region, where t is the time of exposure and p a constant, the value of which varies from 0·8 to 0·9, depending on the plate employed. For intermittent X-rays, as in the present work, the value of p is 1 exactly[10a]. In order to determine the value of "p" for soft X-rays, we give a number of trial exposures by varying the electronic current in the X-ray tube and keeping the time of exposure the same. The intensity of soft X-rays is found to be directly proportional to the electronic current [11].

Fig. 5 is a plot of density against logarithm of intensity of soft X-rays. Points marked with circles were obtained with time exposures, while points marked with crosses were obtained by varying electronic current. It will be seen that both sets of points lie on the same curve, from which we have concluded that the value of "p" does not differ from unity within the source of experimental error. It is, however, found difficult to secure steadiness when electronic current is changed, and hence we gave

time exposures for intensity measurements. Our results in this connexion go to show that for Golden Isozenith plates, developed according to the specification already mentioned, density is directly proportional to intensity up to a density equal to 0·25, while it is proportional to the logarithm of intensity for densities from 0·25 to 1·50. Still larger densities cannot be measured accurately with a microphotometer, and were not therefore admitted for measurements. For X-ray films having coatings on both sides density has been found [12]

Fig. 5.



to be proportional to the intensity over a wider range of densities.

VI. *Determinations of Wave-lengths.*

The wave-lengths for which absorption is determined are W M_α, K-absorption edge in Al, L_α-lines of copper, nickel, cobalt, iron, manganese, and chromium, and K_α-line of oxygen.

The wave-lengths of L_α-lines as measured by different experimenters [13] differ considerably among themselves.

The mean values have been taken. For tungsten M_α-line the crystal value after Lindberg [14] has been increased by 0·22 per cent., while for oxygen K_α-line Soderman's value [15] has been increased by 0·22 per cent. Similar changes are made for other crystal values. The reason of this will be understood from the accurate measurements due to Bearden and Bäcklin [16].

TABLE I.

Mass Absorption Coefficient $\frac{\mu}{\rho}$ in Aluminium.

Lines.	Wave-length in Ångströms.	$\frac{\mu}{\rho}$ (Jönsson).	$\frac{\mu}{\rho}$ (Allen).	$\frac{\mu}{\rho}$ (Authors).
W M_α	6·984	2800	—	2900
Al K abs. edge	7·950	—	*{ 3600 280	*{ 4300 304
Al K_α	8·340	344	330	—
Mg K_α	9·890	553	500	—
Na K_α	12·12	767	850	—
Cu L_α	13·43	—	—	1040
Ni L_α	14·56	—	—	1740
Co L_α	15·96	—	—	2330
Fe L_α	17·70	—	—	3070
Mn L_α	19·55	—	—	4050
Cr L_α	21·73	—	—	5430
O K_α	23·68	—	—	6920

* Of the two values the first one refers to the absorption coefficient on the short wave-length side and the second one to the long wave-length side of the K-absorption edge in aluminium.

VII. *Experimental Results of the Authors.*

The mass absorption coefficients are tabulated in Table I. Some of the values due to Jönsson [4] and also the values compiled by Allen [17] are inserted.

VIII. *Discussions on Experimental Results and Concluding Remarks.*

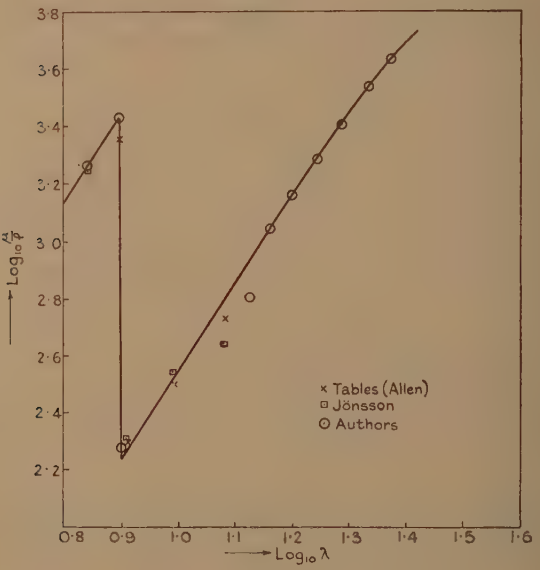
Let us take the formula proposed by Walter [18] from an analysis of experimental data on the absorption coefficients in the ordinary X-ray region. According

to Walter for X-ray wave having a length intermediate between K and L absorption edges, the atomic absorption can be represented by

$$T_\alpha = 8{\cdot}52 \times 10^{-28} z^{4{\cdot}30} \lambda^3.$$

In the present investigation such a simplified law is found to be violated, because a plot of the logarithm of absorption coefficient against wave-length gives a curve

Fig. 6.



(fig. 6), the slope of which decreases slightly as the wave-length increases. A similar type of result has also been obtained by Jönsson for waves having lengths up to 12 Å. So it would appear that the mass absorption coefficient for an absorber can be regarded as $c\lambda^n$ where n is not a constant.

For ordinary X-rays the value of n is 3. As we pass to longer wave-lengths the value of n begins decreasing. For aluminium from 8 to 25 Å. an average value of $n=2{\cdot}95$ may be attributed to represent experimental results to a first order of approximation.

It will be interesting to compare our results with the values expected from Jönsson's [4] universal absorption curve. From our data the absorption per electron $(\mu_e)_K$ has been calculated according to the method outlined by Jönsson. These values, as also those from the so-called universal absorption curve, are listed side by side in Table II., in which z and N represent atomic number and Avogadro's number respectively.

It will be seen that the electronic absorption coefficients computed from the present data for aluminium differ systematically from those obtained from Jönsson's curve. Similar systematic deviations are also to be observed in the experiments of Woernle [6], even from 2 Å. to 10 Å. This points out clearly that Jönsson's

TABLE II.

$\text{Log}_{10}(z\lambda)$.	$\text{Log}_{10}[(\mu_e)_K \times N]$ (due to authors).	$\text{Log}_{10}[(\mu_e)_K \times N]$ (due to Jönsson).	Difference.
2·20	4·065	4·38	+0·32
2·30	4·385	4·60	+0·21
2·40	4·71	4·84	+0·13
2·50	4·98	5·04	+0·06

curve may be regarded as giving rough estimates of absorption coefficients.

The phenomenon of X-ray absorption has also been treated according to Quantum mechanics. Stobbe [19] has deduced expressions for absorptions by the electrons in different shells. For absorption due to electrons in the L-levels for wave-length much shorter than the L-absorption edges, the atomic absorption can be represented by

$$T_\alpha = 5{\cdot}98 z^5 \lambda^{7/2} + 9{\cdot}84 \times 10^5 z^7 \lambda^{9/2}.$$

For low atomic absorber like Al, an evaluation of the effective atomic number presents difficulty because of the uncertainty in the magnitude of the screening constant.

At 10 Å. the atomic absorption from our curve (fig. 6) is equal to $2{\cdot}34 \times 10^{-20}$. This experimental value will be obtained if we take the effective value of the atomic

number to be 6·0 arbitrarily. But then, the theoretical values at 16, 20, and 24 Å. become 19·8, 36·3, and 76×10^{20}, while the experimental ones are 9·55, 19·5, and $33{\cdot}1 \times 10^{20}$ respectively. The agreement is thus inadequate, and the formula gives too large values of absorption with increasing wave-lengths. Discordant results of a similar type are also to be noticed for L-absorption in tin [19], [20].

In the soft X-ray region the rôle played by electrons close to the optical levels becomes appreciable, and hence the simple laws holding good in the ordinary X-ray region for electrons in the deeper levels begin to turn to complications.

In conclusion, we wish to express our deep sense of gratitude to the Director of Public Instruction, Bihar and Orissa (India), for the grant of a research scholarship to one of us (A. T. M.).

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LXXII. *On the Steady Two-Dimensional Motion of Ground-Water with a Free Surface.* *By* B. Davison *.

Introduction.

THE present article is devoted to the examination of steady two-dimensional motion of ground-waters in a uniform permeable ground, the region of motion being supposed to be bound by impervious surfaces, water-basins, surfaces where water oozes out, and free surfaces. The boundaries of the region of motion corresponding to the impervious surfaces, water-basins, and the seepage surfaces are supposed to be formed by straight line segments.

The following cases may be taken as examples illustrating the motion in question :—Percolation through a uniform earth dam of a polygonal cross-section ; ground-water motion in the vicinity of a drain or irrigation canal or pipe also of a polygonal cross-section, or a system of such pipes or canals provided they are parallel to each other.

As a starting-point the author takes the differential equations of ground-water motion given by Forchheimer †, and independently by Slichter ‡ in accordance with the experimental investigations by Darcy §.

The author of the present article gives a method of solution of these equations and determines whether the solution obtained is general, the boundary conditions being as outlined above ||.

The characteristics of motion to which these equations correspond have been outlined by Boussinesq ¶ and worked out in detail by Pavlovsky **.

* Communicated by Prof. N. S. Koshliakov.

† " Ueber die Ergiebigkeit von Brunnen-Anlagen," *Z. des Arch- und Ing.-Vereins zu Hannover*, pp. 541–547 (1886).

‡ " Theoretical Investigation on the Motion of Ground-Waters." XIX. Annual Report U.S. Geolog. Survey, 1897–98, part ii.

§ ' Les fontaines publiques de la ville de Dijon,' Paris, 1856, p. 590, ff.

|| A solution of a particular case of this problem which had been published by the author in 1932 (Davison, " On the Steady Motion of Ground-Water through the Earth Dam," ' Memoires de l'Institute Hydrologique,' vol. vi. (in Russian)) was given by G. Hamel (*Zeitschrift für Angewandte Mathematik und Mechanik*, 1934, p. 3).

¶ " Recherches théoriques sur l'écoulement des nappes d'eaux infiltrées dans la sol, "*Journal de Mathématique pure et appliquée* (*Journ. de Liouville*), 1904, pp. 11–16; ibid. "Complètement . . . ," pp. 364–371.

** ' On the Ground-Water Motion under the Hydro technical Structures ' (in Russian).

In § 1 are given the basic assumptions established in the main by Boussinesq and Pavlovsky and which we give in a slightly modified form.

In § 2 is given the deduction of the differential equations, and it is shown that the solution is analytic at all points within the region.

In § 3 the boundary conditions are established, and the question whether the solution is analytical at the points on the boundary is discussed. In § 4 the method of solution is given.

The reasonings given below indicate that any solution satisfying the fundamental postulates may be obtained by the method proposed by the author.

In another paper, "On the Steady Motion of Ground-Water through a Wide Permeable Prismatic Earth Dam," the author, examining but one particular class of domains, shows that the solution depends on exactly as many parameters as must be given in the data of the problem.

This leads us to expect that the solution is the only possible one.

At the end of the present paper an example is given where the method is applied to the percolation of water through a permeable dam of a rectangular cross-section, this dam being supported on a horizontal impervious base. Some of the conclusions arrived at in the above-mentioned second paper by the present writer are made use of in the working out of this problem.

§ 1. *Basic Assumptions.*

I. "The pores of a soil are arranged in such a haphazard manner, that we are at liberty to suppose that at any particular point the fluid is free to move in any desired direction." *

That is why it is both purposeless and impossible to examine an actual fluid particle moving in the pores of the soil. We have, on the contrary, to examine a, so to say, principal motion which we may regard as being the motion of some imaginary liquid, neglecting "the pulsations" of velocities and without considering whether a given point is in reality occupied by water or by a soil particle.

That is why it appears to be more to the purpose to consider not the average velocity itself, but the so-called

* C. Slichter, *op. cit.* p. 330.

percolation velocity, *i. e.*, the ratio of the flux through a surface element, normal to the direction of average velocity, to the area of this element.

Evidently the percolation velocity is the average velocity multiplied by the coefficient of porosity. The percolation velocity may be regarded as the velocity of the above-mentioned imaginary fluid.

II. In studying the motion of this imaginary fluid the resistance is to be regarded as a volume force, because in actual conditions the soil particles which cause the friction can be found within any volume.

II. *a.* In accordance with the experiments of Darcy * we assume that the resistance is proportional to the first power of the percolation velocity, and the coefficient of proportionality we regard as having the same value throughout the region of motion.

III. Since, on the one hand, the magnitudes of the inertia forces for the particles of an actual liquid may vary very rapidly, and, on the other hand, their expressions in terms of velocities of actual particles and the derivatives of these velocities are not linear, the inertia forces of a particle of the imaginary fluid can by no means express the resultant of the inertia forces of the actual particles. This resultant cannot be calculated as we do not know the arrangement of the soil pores, but we have a right to expect that it brings about a certain increase of the resistance.

III. *a.* In any case the "inertia forces" of the imaginary fluid can be, generally speaking, but very small as compared with the percolation velocity; because this velocity itself, as well as its derivatives, is very small in ordinary conditions; and the "inertia forces" in terms of this velocity are expressed by a quadratic polynomial.

Bearing in mind the above statement we shall neglect the inertia forces of the imaginary fluid and take into account the inertia forces of the actual particles only by

* A number of other investigators have come to the same conclusions as Darcy. Some of them, however (as, *e. g.*, Masoni, 'Sul moto dell' aqua attraverso i terrini permeabile' and Baldwin-Wiseman, Min. of Proc. Eng. 1910, vol. clxxxi. pp. 15–61), obtain slightly different results. The theoretical explanation of Darcy's results had been indicated by Dupuit, and was worked out in detail by Pavlovsky, who regarded the pores as capillary tubes.

slightly increasing the resistance, which we shall do in accordance with the assumptions II. and II. *a*.

IV. We shall not examine the actual pressure, which depends on the distribution of the pores in the soil, but a certain average pressure corresponding to the above-mentioned generalized motion.

V. Since the adjacent pores are connected with each other the actual pressure will vary gradually, and therefore we shall regard the above-mentioned average pressure as being continuous throughout the region of motion including the boundary, and hence as being bounded in any bounded part of the region.

IV. The pressure and the component velocities are single valued functions of the coordinates. (The necessity of this condition is obvious.)

VII. To have a right to use the differential equations we must suppose that the partial derivatives of the first order of the average pressure and of the percolation velocity components exist and are continuous within the region of motion with the exception, maybe, of some points and lines. Points and lines where the above partial derivatives have a discontinuity or do not exist we shall call singular points.

VIII. Since the soil particles and the pores between them have finite dimensions it would not be logical to suppose that the distance between two singular points can be infinitesimal. Therefore every singular point we shall regard as an isolated singular poiut. On the same grounds we have to regard every singular line as rectifiable and accessible at any of its points from both sides *.

IX. Since the liquid can neither appear nor disappear, any sources and sinks (both point sources and continuously distributed sources as well as any other) within the region are excluded.

In accordance with the nature of the problem we could have as sources and sinks only different hydrotechnical constructions such as, for instance, underground draining

* The above reasonings permit further to restrict the singular lines (in the solution of our problem), which will at once become evident if we imagine a non-rectifiable curve. We do not, however, need that, and therefore we do not introduce any further restrictions.

galleries, but such constructions are given when the problem is brought up.

Therefore they evidently must not be considered as belonging to the region of motion.

IX. *a*. The above-mentioned hydrotechnical constructions will be regarded as possessing a finite area and not as point sources, because otherwise the pressure value at these points would be infinity, which would contradict the assumption V. *.

X. The flux through any arc of any rectifiable curve which belongs to the region of motion (and which can have common points and arcs with the contour of the region) must have a definite finite value and must be a continuous function of the end points of the arc.

X. *a*. It may happen, however, that for the determination of the above-mentioned finite flux-value in all cases it will not be sufficient to have the above assumptions. On the other hand, it is known that for any arc on which there are no singular points and which has no common points with singular lines the flux is given by the integral

$$\int_L \rho(v_x\,dy - v_y\,dx).$$

Therefore we shall further assume, in addition to postulate X., that whatever is the arc, even, for instance, in the case when it is an arc of a singular line, the above-mentioned finite value of the flux is given by the above integral. Thereby, this integral must exist whatever integration path is chosen.

XI. We shall regard the fluid as incompressible, *i. e.*, we shall assume the density to be constant ($\rho = \text{const.}$).

XII. The motion will be regarded as two dimensional.

XIII. Steady motion will be considered.

The impossibility of working out a theory of ground-water motion without introducing most of the above postulates seems to be obvious. The assumptions II. *a*, III. *a*, IX. *a*, XI., XII., and XIII. are introduced partly in accordance with the experimental data and partly for the sake of simplicity.

* We may omit these restrictions, but then the assumptions V, X. and X. *a*, as well as some of our conclusions, would have to be formulated in a somewhat more complicated way.

§2. *Differential Equations and the Character of Motion.*

Since, in accordance with the assumption III. *a*, we shall not take into account the acceleration of the imaginary fluid, the dynamical equations must show that the resultant of all forces applied to a volume must be zero. These forces are gravity, pressure, and resistance. Calculating the magnitudes of these forces and putting their resultant $=0$, we have

$$-\frac{\partial p}{\partial x}-\frac{\rho g}{k}v_x=0\,; \quad -\rho g-\frac{\partial p}{\partial x}-\frac{\rho g}{k}v_y=0,$$

or, which is the same

$$v_x=-k\frac{\partial}{\partial x}\left(\frac{p}{\rho g}+y\right); \quad v_y=-k\frac{\partial}{\partial y}\left(\frac{p}{\rho g}+y\right), \quad . \quad (1)$$

where the Y-axis is directed vertically upward, the x-axis is horizontal, v_x and v_y are the component velocities; p=the average pressure; ρ=density; g=the acceleration due to gravity; and k=the percolation coefficient.

The continuity equation may be written in its usual form

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equations (1) show that there is a velocity potential, viz.,

$$\phi=-k\left(\frac{p}{\rho g}+y\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\left(v_x=\frac{\partial \phi}{\partial x}; \quad v_y=\frac{\partial \phi}{\partial y}\right).$$

Therefore the equation (2) may be rewritten in the form

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On the other hand, it follows from the assumption VII. that the derivatives of ϕ of the first and of the second order exist, and are continuous.

Therefore, ϕ is a harmonic function, and, in accordance with a well-known theorem, its derivatives of any order exist and are continuous with the possible exception

of the above-mentioned singular points and singular lines (see assumption VII.). According to V the pressure p is continuous throughout the region including the boundary. It means that the velocity potential ϕ is also continuous throughout the region including the boundary. Moreover, p being single-valued, ϕ is single-valued too. On the other hand, in accordance with the assumption IX. the flux throughout any arc MM_1 in any singly connected part of the region depends only on the end points of this arc, *i.e.*, on the points M and M_1, but, in accordance with X. *a*, the flux is a continuous function of these ends and can be represented by the integral

$$\int_{M M_1} \rho(v_x dy - v_y dx),$$

even in the case when this arc is an arc of a singular line. Therefore this integral depends only on the position of the points M and M_1, and is a continuous function of these points throughout the region including the boundary, unless multiply connected regions are examined.

In accordance with XI. ρ is constant, and therefore the above holds good also for the integral

$$\psi = \int_{M M_1} (v_x dy - v_y dx).$$

This integral, obviously, is the conjugate harmonic function of the velocity potential ϕ (ψ is usually called the stream function).

Thus the complex quantity $f = \phi + i\psi$ (which may be denoted by the term complex potential) is an analytic function of the complex variable $z = x + iy$ continuous throughout the region of motion, including the boundary. As a continuous function f is bounded if the region of motion is bounded, and if the region of motion extends to infinity f is bounded in any bounded part of the region. Moreover, as has been shown above, the stream-function ψ is single-valued in any singly-connected part of the region. The velocity-potential is, as we have seen, single-valued throughout the region of motion. Thus the complex potential f is also single-valued in any singly-connected part of the region of motion.

Therefore the complex potential f cannot have any isolated singular points belonging to the region of motion.

In fact the only isolated singular points of a single-valued analytic function are poles and essential singularities, and, on the other hand, an analytic function bounded in a region can have in this region neither poles nor essential singularities.

The above statements lead us to conclude that no vortex can exist in the region of motion of ground-waters because an isolated singular point of the complex potential f always corresponds to a vortex.

Let us, moreover, show that the complex potential cannot have any singular lines within the region. In accordance with the assumption VIII. any singular line is rectifiable and is the common boundary of two open regions within both of which f is analytic.

On the singular line itself f, as we have seen, is bounded and continuous. Therefore, as it is not difficult to show, making use of the Cauchy's integral *, the complex potential must be analytic on the line in question, too, and its values on one side of this line must be an analytic continuation of its values on the other side. The above reasonings show that the so-called discontinuity lines are absent in ground-waters, while these lines are, on the contrary, possible in an ideal frictionless incompressible liquid.

§ 3. *Boundary Conditions.*

Let us now pass to the examination of the boundary of the region of motion. This boundary can be formed by

(1) Water-basins.
(2) Impervious surfaces.
(3) Separate areas.
(4) Free surfaces (inside the permeable ground).

As has been pointed out at the very beginning, we assume that the boundaries of water-basins, impermeable surfaces, and seepage surfaces are formed by segments of straight lines.

Since the percolation flux is very small in comparison with the boundary area of water-basins, the velocities in the basins must be very small, too, and therefore the

* Since f is continuous, Cauchy's integral can be applied to any region bounded by a simple rectifiable contour inside which f is analytic. See, for instance, Smirnoff "Sur la théorie des polynomes orthogonaux à une variable complex," *Bulletin de Soc. Phys.-Mathem. à Leningrad*, 1928, p. 159.

values of pressure near these boundaries cannot appreciably differ from the pressure values given by the hydrostatic law. Therefore, as a first approximation, we shall assume that the pressure at the boundaries of the water-basins obeys the hydrostatic law.

This assumption agrees sufficiently well with the experimental data. In a mathematical form it will be written as

$$p=p_0+\rho g(h-y), \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where p_0 is the pressure on the upper surface of the water-basin, which is usually equal to the atmospheric pressure, and h is the height of the water-level measured from the axis of abscissa. Putting together assumption (5) and equation (3) we have

$$\phi=-k\left\{\frac{1}{\rho g}[p_0+\rho g(h-y)]+y\right\}=-k\left(\frac{p_0}{\rho g}+h\right)$$

or

$$\phi=\text{const.}$$

That is, the boundary of any water-basin is an equipotential line.

No flux can pass through an impervious surface, and therefore the integral

$$\int(v_x dy-v_y dx)$$

taken along any part of such a boundary is equal to zero.

Thus we see that on these parts of the boundary the stream function has a constant value.

On the seepage areas the pressure will be assumed to be constant and equal to atmospheric pressure (obviously the second part of this assumption is but a corollary of the supposition that the pressure is continuous).

Therefore for these parts of the boundary the equation (3) will give

$$\phi+ky=-k\frac{p_0}{\rho g}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let us further assume that the pressure on the free surface is also constant. Then it is easy to see that the equation (6) holds good for this surface.

All the parts of the boundary with the exception of free surfaces are given, and, as has already been pointed out, they are formed by straight line segments.

Thus for these parts of the boundary we have

$$ax+by=c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Let us now examine the free surface. If we neglect the evaporation and the penetration into the ground of rain-water, then, since we are considering a steady motion, the flux across a free surface will be equal to zero, and on the free surface we shall have

$$\psi=\text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the evaporation and the penetration of rain-water into the ground are to be taken into account, then, as a first approximation, we may suppose that the evaporation from any arc of the free surface is proportional to the projection of this arc upon a horizontal axis, and a like assumption may be made with respect to the penetration of rain-water into the soil *.

Therefore we may write

$$\psi-\epsilon x=\text{const.},$$

where ϵ is a constant (which may be different in different cases), positive when the penetration is greater than evaporation and negative when, on the contrary, the evaporation is greater. In order to compare the constants k and ϵ, let us point out that k, as we can see from equation (1), is the absolute value of the percolation velocity in case of the pressure being uniform throughout the region.

This leads us to suppose that the absolute value of ϵ is considerably less than that of k. Thus we see that for every portion of the boundary we have two equations of the form

$$\left.\begin{aligned} a_j x+b_j y+c_j\phi+d_j\psi &= k_j, \\ a_j' x+b_j' y+c_j'\phi+d_j'\psi &= k_j'. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

These equations may also be written in the form

$$\left.\begin{aligned} \mathrm{I}(m_j z+n_j f) &= k_j, \\ \mathrm{I}(m_j' z+n_j' f) &= k_j', \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (10)$$

where I denotes the imaginary part of the complex quantity ; m_j, n_j, m_j', n_j' are constants.

* The evaporation from a free surface and the penetration of rain-water depend on so many factors that if we assumed them to be proportional to the length of the arc of the free surface we probably should not in any way approach the real conditions, but, maybe, even should get further away from them.

Moreover, for every portion of the boundary we have the inequality

$$m_j n_j' - m_j' n_j \neq 0. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In fact, for seepage areas, water-basin boundaries, and impervious surfaces one of the equations (10) contains only the coordinates (x, y), whilst the other of these equations must necessarily contain either ϕ or ψ, and, therefore, we may, for instance, assume

$$m_j \neq 0 \; ; \; n_j = 0 \; ; \quad n_j' \neq 0.$$

For the free surface the equations (10) become

$$\left.\begin{aligned} \mathrm{I}(if + kz) &= k_j, \\ \mathrm{I}(f - i\epsilon z) &= k_j', \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10')$$

and the inequality (11) is evidently satisfied, provided only that ϵ is not equal to k.

But, as has been pointed out above, the absolute value of ϵ is considerably less than that of k, so that the equality $k = \epsilon$ is excluded.

(It may also be pointed out that, if ϵ was equal to k, the equations (10) would give for the free surface $f = ikz + \text{const.}$, but such an equality being satisfied on a line, it must be satisfied in the whole of the region, which would contradict the conditions on other parts of the boundary.)

It has been shown in § 2 that the solution is analytic in all points within the region of motion. Equations (10) (the inequality (11) being satisfied) permit us to find out whether the solution is analytic also on the boundary.

For this purpose let us introduce an auxiliary complex variable $\zeta = re^{i\vartheta}$, where $\zeta(z)$ is a function giving a conformal representation of the region of motion on the circle $|\zeta| = r \leq 1$. Then in the plane ζ an arc of the circumference corresponds to every portion of the boundary. Let us, for instance, examine the jth portion of the boundary. Equations (10) show that the corresponding arc of the circumference in the plane ζ is, in the planes $m_j z + n_j f$ and $m_j' z + n_j' f$, transformed into straight line segments, the above arc being all the time traversed in one direction. As to z and f, z can be infinite only at the ends of the portion considered, and the same holds good with respect to f, since f is a continuous function of z; thus neither $m_j z + n_j f$ nor $m_j' z + n_j' f$ can be infinite within the portion

in question. Bearing in mind the above considerations, we may apply Schwarz's principle of symmetry, and then we come to the conclusion that the functions $m_j z+n_j f$ and $m_j' z+n_j' f$ are analytic in the jth portion of the boundary, and can be continued beyond it. The same holds good for the expressions

$$n_j'(m_j z+n_j f)-n_j(m_j' z+n_j' f)=z(m_j n_j'-n_j m_j'),$$
$$m_j'(m_j z+n_j f)-m_j(m_j' z+n_j' f)=-f(m_j n_j'-n_j m_j');$$

that is, for z and f themselves, because the determinant $(m_j n_j'-m_j' n_j)$ is not equal to zero. Since j is arbitrary, all the above may be applied to any portion of the boundary, and the only possible singular points of the functions $z(\zeta)$ and $f(\zeta)$ are the end-points.

§ 4. *The Method of Solution.*

The boundary conditions show that neither the region of z nor that of f can be plotted. The equations (10), however, allow us to plot the region of

$$w=v_x-iv_y=\frac{df}{dz}.$$

In fact, both z and f are analytic functions of $\zeta=re^{i\vartheta}$, regular within every portion of the boundary.

On the boundary $r=|\zeta|$ is constant, viz., $r=1$ whilst $\vartheta=\arg\zeta$ is variable. Therefore we may differentiate both equations (10) with respect to ϑ, and then we shall get

$$\frac{\partial \mathrm{I}(m_j z+n_j f)}{\partial \vartheta}=0\,;\quad \frac{\partial \mathrm{I}(m_j' z+n_j' f)}{\partial \vartheta}=0. \quad . \quad . \quad (12)$$

The second of the equations (12) may be written in the form

$$\frac{\partial \mathrm{I}(m_j' z+n_j' f)}{\partial \mathrm{R}(m_j z+n_j f)}\cdot\frac{\partial \mathrm{R}(m_j z+n_j f)}{\partial \vartheta}$$
$$+\frac{\partial \mathrm{I}(m_j' z+n_j' f)}{\partial \mathrm{I}(m_j z+n_j f)}\cdot\frac{\partial \mathrm{I}(m_j z+n_j f)}{\partial \vartheta}=0. \quad . \quad (12')$$

The first of the equations (12) shows that the second term in the left side of equation (12′) is equal to zero. Therefore, on the jth portion of the boundary we have

$$0=\frac{\partial \mathrm{I}(m_j'z+n_j'f)}{\partial \mathrm{R}(m_jz+n_jf)}=\mathrm{I}\left(\frac{d(m_j'z+n_j'f)}{d(m_jz+n_jf)}\right)=\mathrm{I}\left(\frac{m_j'+n_j'\dfrac{df}{dz}}{m_j+n_j\dfrac{df}{dz}}\right)$$

or

$$\mathrm{I}\left(\frac{m_j'+n_j'w}{m_j+n_jw}\right)=0. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This holds good for all points except those where $m_j\dfrac{dz}{d\zeta}+n_j\dfrac{df}{d\zeta}$ is equal to zero, *i. e.*, except a few separate points. On the other hand, since

$$\frac{m_j'+n_j'w}{m_j+n_jw}=\frac{d(m_j'z+n_j'f)}{d\zeta}:\frac{d(m_jz+n_jf)}{d\zeta}$$

is an analytic function of ζ, the boundary of the region of $\dfrac{m_j'+n_j'w}{m+nw}$ must be continuous with the exception of those points where $\dfrac{m_j'+n_j'w}{m_j+n_jw}$ is infinite. Therefore equation (13) holds good for all points of the jth portion of the boundary except those where the above ratio is equal to infinity.

Equation (13) shows that in the $\dfrac{m_j'+n_j'w}{m_j+n_jw}$ plane a straight line segment corresponds to the jth portion of the boundary; consequently in the w-plane we have either an arc of a circumference or a straight line segment, viz., a straight line segment if $n_j=0$, $n_j'=0$, or if the ratio $\dfrac{n_j'}{n}$ is real, and an arc of a circumference of a finite radius in all other cases.

Thus we see that the equations (13) enable us to plot all the lines of which the arcs form the boundary of the region w; and, properly choosing these arcs, we can determine the w-region itself.

Now, since f and z are analytic functions of ζ regular at all points of the boundary except the end-points of its portions, $w=\dfrac{df}{d\zeta}:\dfrac{dz}{d\zeta}$ must also be regular again with

the exception of the above-mentioned end-points, and, maybe, the points where $\frac{dz}{d\zeta}$ is equal to zero. In the latter points, *i. e.*, where $\frac{dz}{d\zeta}=0$, w, within any portion of the boundary, is either analytic or has a pole, but it can have no other singularities, because both $\frac{df}{d\zeta}$ and $\frac{dz}{d\zeta}$ are regular at these points. But, on the water-basin boundaries, impervious surfaces, and seepage areas, $\frac{dz}{d\zeta}$ cannot equal zero, because every one of these portions is traversed in z-plane in a definite direction. On the free surface w is bounded, since on the w-plane a finite radius circumference corresponds to a free surface. Therefore $w(\zeta)$ also cannot have any poles on the arc corresponding to a free surface. Thus we come to the conclusion that w must be regular within any portion of the boundary again with the exception of the end-points of the portions.

Therefore we may differentiate equation (13), which holds good for the jth arc of the line $r=1$ with respect to ϑ, and we shall obtain

$$\frac{\partial}{\partial\vartheta}\mathrm{I}\left(\frac{m_j'+n_j'w}{m_j+n_jw}\right)=0. \quad . \quad . \quad . \quad (14)$$

Let us now take one of the equations (12), *e. g.*, the first. Since w is regular within the jth portion of the boundary, $\frac{m_j'+n_j'w}{m_j+n_jw}$ must be regular as well, with the exception of the points where $m_j+n_jw=0$; hence the first of the equations (12) may be written in the form

$$\frac{\partial\mathrm{I}(m_jz+n_jf)}{\partial\mathrm{R}\left(\frac{m_j'+n_j'w}{m_j+n_jw}\right)}\cdot\frac{\partial\mathrm{R}\left(\frac{m_j'+n_j'w}{m_j+n_jw}\right)}{\partial\vartheta}$$

$$+\frac{\partial\mathrm{I}(m_jz+n_jf)}{\partial\mathrm{I}\left(\frac{m_j'z+n_j'w}{m_j+n_jw}\right)}\cdot\frac{\partial\mathrm{I}\left(\frac{m_j'+n_j'w}{m_j+n_jw}\right)}{\partial\vartheta}=0.$$

Combining this with equation (14), we have

$$0=\frac{\partial \mathrm{I}(m_j z+n_j f)}{\partial \mathrm{R}\left(\frac{m_j'+n_j' w}{m_j+n_j w}\right)}=\mathrm{I}\left(\frac{d(m_j z+n_j f)}{d\left(\frac{m_j'+n_j' w}{m_j+n_j w}\right)}\right)$$

$$=\mathrm{I}\left(\frac{(m_j+n_j w)^3}{m_j n_j'-m_j' n_j}\cdot\frac{dz}{dw}\right),$$

and, therefore,

$$\arg\frac{dz}{dw}+3\arg(m_j+n_j w)-\arg(m_j n_j'-m_j' n_j)=n\pi, \quad (15)$$

where n is an integer which, within the portion, can vary only at the points to which the deduction of the above formula cannot be applied. This deduction cannot be applied only

(1) At the points where $\frac{m_j'+n_j' w}{m_j+n_j w}$ has a pole.

(2) At the points where $\frac{d}{d\bar{\zeta}}\left(\frac{m_j'+n_j' w}{m_j+n_j w}\right)$ equals zero; and

(3) At the points where $(m_j+n_j w)^3\frac{dz}{dw}$ equals zero or infinity.

But $\frac{m_j'+n_j' w}{m_j+n_j w}$ can have a pole only where $m_j+n_j w$ equals zero, because $m_j'+n_j' w$ cannot equal infinity within the portion, $\frac{d}{d\bar{\zeta}}\left(\frac{m_j'+n_j' w}{m_j+n_j w}\right)$ equals zero only where $\frac{dw}{d\zeta}$ equals zero, since $m_j+n_j w$ cannot equal infinity within the portion; and $\frac{dz}{dw}$ can, within the portion, equal zero or infinity only at the points where either $\frac{dw}{d\zeta}$ or $\frac{dz}{d\bar{\zeta}}$ equals zero. Therefore the integer n in formula (15) can vary within the jth portion of the boundary only at the points where either $m_j+n_j w$, $\frac{dw}{d\zeta}$, or $\frac{dz}{d\bar{\zeta}}$ equal zero.

On the portion of the boundary with which we shall begin we may evidently ascribe to n any integer value; but in order to determine n for the end-points, and for those points to which formula (15) cannot be applied, we must examine the increments of arg dz and arg dw at the passage of these points. In this way the values of the harmonic function $\arg \frac{dz}{dw}$ will be determined for the whole boundary of the w-region. After that the solution may be carried out as follows :—

Equations (13) enable us to plot the w-region, and, knowing this region, we can plot the analytic function $w(\zeta)$, which gives us a conformal representation of ζ-region on the w-region *.

This will permit us to compute the values of $\arg \frac{dz}{d\bar{w}}$ on the boundary of the ζ-region, knowing those of $\arg \frac{dz}{d\bar{w}}$ on the boundary of w-region; and, moreover, it will enable us to determine the harmonic function $\arg \frac{dw}{d\bar{\zeta}}$. Thus we shall obtain the boundary values of the harmonic function

$$\arg \frac{dz}{d\bar{\zeta}} = \arg \frac{dz}{dw} + \arg \frac{dw}{d\bar{\zeta}}.$$

If z-region is singly connected, then, returning to the starting-point on the boundary, we shall also, obviously, have the initial value of $\arg \frac{dz}{d\bar{\zeta}}$. The function, $z(\zeta)$, being a conformal representation of ζ-region on the region of motion, must obviously be regular at any point within the region as well as the inverse function $\zeta(z)$. The same is true with respect to their derivatives. Therefore $\frac{dz}{d\bar{\zeta}}$ is regular, and is not equal to zero. Hence $\log \frac{dz}{d\bar{\zeta}}$ is a regular analytic function, and its imaginary part,

$$\mathrm{I}\left(\log \frac{dz}{d\bar{\zeta}}\right) = \arg \frac{dz}{d\bar{\zeta}},$$

* Conformal representation of a region bounded by arcs of circumferences and by straight lines can be reduced to integration of an ordinary differential equation depending on certain parameters. See Hurwitz-Courant, ' Functionen theorie,' p. 410.

is a regular harmonic function. The boundary values of this function are known. ζ-region is a circle, and, therefore, we can obtain the values of this harmonic function at any point within the region by making use of Poisson's integral *.

After that we shall be able to determine its conjugate harmonic function, viz., $\log \frac{1}{\left|\frac{dz}{d\zeta}\right|}$, but this will introduce an arbitrary constant, so that $\frac{dz}{d\zeta}$ and, consequently, z will contain an arbitrary real factor. Determining the function $\zeta(z)$ inverse to the function $z(\zeta)$ that we have obtained, and substituting $\zeta(z)$ in $w(\zeta)$, we shall have the final solution of the problem.

5. *Solution for a Dam of a Rectangular Cross-section on a Horizontal Impervious Base* †.

Let us apply the above results to a problem of percolation of ground-water through a dam of a rectangular cross-section on a horizontal impervious basis. We shall neglect the effects of evaporation of water.

Let us take the upper-water boundary for the axis of ordinates and the base of the dam for the abscissæ axis, directing the latter towards the tail-water. Let l denote the length of the base, h_1 the upper-water depth, and h_2 tail-water depth. The upper-water boundary we shall call the first portion of the contour, the base the second portion etc.

Then for the first portion we shall have

$$x=0\,; \quad \phi=-k\left(\frac{p_0}{\rho g}+h_1\right);$$

and in formula (10) we may put

$$m_1=i\,; \quad n_1=0\,; \quad m_1'=0\,; \quad n_1'=i.$$

* In another paper, "On the Steady Motion of Ground-Water through a Wide Prismatic Permeable Dam," the author, taking a particular class of regions, shows that it must be possible to represent $\log \frac{dz}{d\zeta}$ by Poisson's integral, and, therefore, no other function possessing the same boundary values will give the solution of the problem.

† Davison "On the Steady Motion of Ground-Water through an Earth Dam"; G. Hamel, 'Ueber Grund wasserströmung' (see footnote, Introduction).

The equation (13) will give $\mathrm{I}\left(\frac{iw}{i}\right)=\mathrm{I}(w)=0$.

For the second portion we shall have

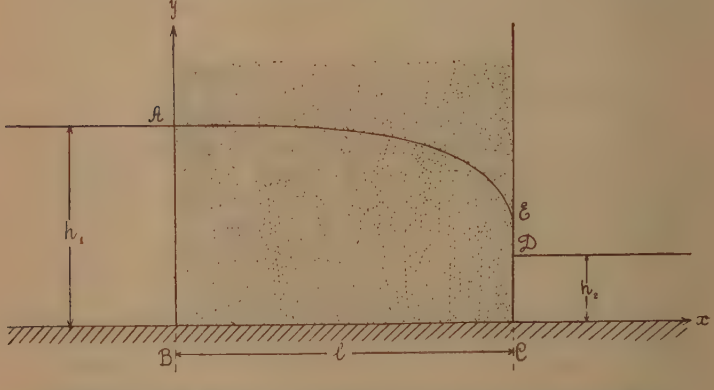
$$m_2=1\,;\quad n_2=0\,;\quad m_2'=0\,;\quad n_2'=1\,;$$

and the equation (13) will again give $\mathrm{I}(w)=0$.

In exactly the same way we shall obtain for the tail-water boundary $\mathrm{I}(w)=0$, for the seepage area $\mathrm{I}(w)=k$, and for the free surface the equation (10) will take the form

$$\mathrm{I}(if+kz)=\frac{-kp_0}{\rho g}\,;\quad \mathrm{I}(f)=q$$

Fig. 1.



(where q is the flux) ; so that the equation (13) will give

$$\mathrm{I}\left(\frac{k+iw}{w}\right)=0,$$

which is the equation of a circumference tangent to the lines $\mathrm{I}(w)=0$ and $\mathrm{I}(w)=k$ at the points $w=0$ and $w=ik$.

In another paper, "On the Steady Motion of Ground-Water through a Wide Impervious Prismatic Earth Dam," we show that the horizontal component of the velocity cannot be negative. It is also shown there that in case of such a dam a seepage surface is unavoidable. Therefore the boundary of the w-region can only be composed of lines

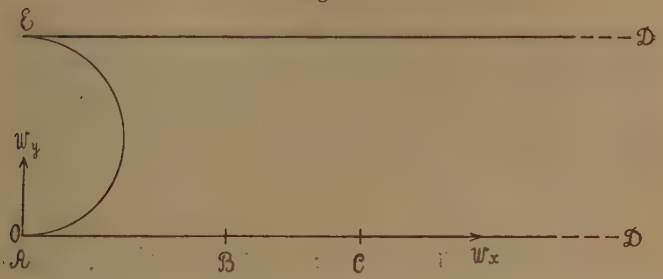
$$\mathrm{I}(w)=0,\quad \mathrm{R}(w)\geqslant 0\,;$$
$$\mathrm{I}(w)=k,\quad \mathrm{R}(w)\geqslant 0\,;$$

and $$\mathrm{I}\left(\frac{k+iw}{w}\right)=0, \quad \mathrm{R}(w)\geqslant 0.$$

Moreover, the whole region of motion must lie within the curvilinear triangle formed by the above lines.

No segment of the perimeter of this triangle can form a part of the w-region boundary more than once, because if that was not the case the whole of the half plane $\mathrm{R}(w)<0$ would belong to the region. Hence it is the above curvilinear triangle which is the region of w.

Fig. 2.



Applying formula (15) to the portion of the contour corresponding to the upper-water boundary, and putting $n=0$, we shall have

$$\arg\frac{dz}{dw}=\arg(i.i-0.0)-3\arg(i-0.w)+0.\pi=-\frac{\pi}{2}.$$

Applying further the same formula (15) to the portion of the boundary corresponding to the impervious base, we have

$$\arg\frac{dz}{dw}=n\pi.$$

On passing from the upper-water boundary AB to the base BC (see fig. 1) $\arg dz$ increases by $\frac{\pi}{2}$, and $\arg dw$ remains unchanged. Hence $\arg\frac{dz}{dw}$ increases by $\frac{\pi}{2}$ and becomes equal to zero. Therefore $n=0$, and along the whole of the base we have

$$\arg\frac{dz}{dw}=0.$$

In the same way, for the tail-water boundary

$$\arg \frac{dz}{dw} = \frac{\pi}{2};$$

for the seepage area

$$\arg \frac{dz}{dw} = -\frac{\pi}{2};$$

and for the free surface

$$\arg \frac{dz}{dw} = -3 \arg w.$$

In our case the conformal representation of the region on the half plane is effected by means of the well-known modular elliptic function. This circumstance considerably simplifies the problem of finding the harmonic function $\arg \frac{dz}{dw}$ within the region.

For instance, let

$$w = k\left\{ \mathrm{F}\left(1 - i\frac{1+\zeta}{1-\zeta}\right) : \mathrm{F}\left(i\frac{1+\zeta}{1-\zeta}\right) + i \right\}, \qquad (16)$$

where

$$\mathrm{F}(\lambda) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\lambda \sin^2 \theta}}. \qquad (17)$$

Availing ourselves of the properties of the modular elliptic function, we may write two other expressions for w, namely,

$$w = k\mathrm{F}\left(\frac{1-\zeta}{(1-i)-(1+i)\zeta}\right) : \mathrm{F}\left(\frac{-i(1+\zeta)}{(1-i)-(1+i)\zeta}\right) \qquad (16a)$$

and

$$w = k\mathrm{F}\left(\frac{1-\zeta}{i(1+\zeta)}\right) : \left\{ \mathrm{F}\left(\frac{(1-i)+(1+i)\zeta}{i(1+\zeta)}\right) - i\mathrm{F}\left(\frac{1-\zeta}{i(1+\zeta)}\right) \right\}. \qquad (16b)$$

Let us denote by $g(\vartheta)$ the values of $\arg \frac{dz}{dw}$ on the circumference $|\zeta| = 1$, *i. e.*, let

$g(\vartheta) = -\pi/2$ for $0 < \vartheta < \alpha$,

$$g(\vartheta)=0 \qquad \text{for} \quad \alpha<\vartheta<\beta,$$

$$g(\vartheta)=\pi/2 \qquad \text{for} \quad \beta<\vartheta<\pi,$$

$$g(\vartheta)=-\pi/2 \qquad \text{for} \quad \pi<\vartheta<\tfrac{3}{2}\pi,$$

$$g(\vartheta)=-3\arg\left\{k\mathrm{F}\left(\frac{1-e^{i\vartheta}}{i(1+e^{i\vartheta})}\right) : \left[\mathrm{F}\left(\frac{(i-1)+(i+1)e^{i\vartheta}}{i(1+e^{i\vartheta})}\right)-i\mathrm{F}\left(\frac{1-e^{i\vartheta}}{i(1+e^{i\vartheta})}\right)\right]\right\}$$

$$=-3\tan^{-1}\{\mathrm{F}(-\tan\vartheta/2) : \mathrm{F}(1+\tan\vartheta/2)\}$$

$$\text{for } \tfrac{3}{2}\pi<\vartheta<2\pi,$$

where α and β are two parameters, $e^{i\alpha}$ and $e^{i\beta}$ being the values of ζ corresponding to the end-points of the base.

Formula (16 *a*) shows that these parameters are connected with the velocities at the points mentioned, viz., for the upper-water end of the base we have the formula

$$v_x=k\mathrm{F}\left(\frac{1-e^{i\alpha}}{(1-i)-(1+i)e^{i\alpha}}\right) : \mathrm{F}\left(\frac{-i(1+e^{i\alpha})}{(1-i)-(1+i)e^{i\alpha}}\right)$$

$$=k\mathrm{F}\left(\frac{\tan\alpha/2}{1+\tan\alpha/2}\right) : \mathrm{F}\left(\frac{1}{1+\tan\alpha/2}\right), \quad . \quad (18)$$

and for its other end

$$v_x=k\mathrm{F}\left(\frac{\tan\beta/2}{1+\tan\beta/2}\right) : \mathrm{F}\left(\frac{1}{1+\tan\beta/2}\right). \quad (18\,a)$$

Now the values of the $\arg\frac{dz}{dw}$ may be found by means of Poisson's integral,

$$\arg\frac{dz}{dw}=\frac{1}{2\pi}\int_0^{2\pi} g(\vartheta)\mathrm{R}\left(\frac{e^{i\vartheta}+\zeta}{e^{i\vartheta}-\zeta}\right)d\vartheta$$

$$=\left\{\tfrac{1}{2}\arg\frac{\zeta-1}{\zeta-e^{i\alpha}}+\tfrac{1}{2}\arg(1+e^{i\alpha})\right\}$$

$$+\left\{\tfrac{1}{2}\arg\frac{\zeta+1}{\zeta-e^{i\beta}}+\tfrac{1}{2}\arg(1-e^{i\beta})\right\}$$

$$+\left\{\tfrac{1}{2}\arg\frac{\zeta+1}{\zeta+i}-\frac{\pi}{8}\right\}+\left\{\frac{-3}{2\pi}\int_{3/2\pi}^{2\pi}\mathrm{R}\left(\frac{e^{i\vartheta}+\zeta}{e^{i\vartheta}-\zeta}\right).\right.$$

$$\left.\tan^{-1}[\mathrm{F}(-\tan\vartheta/2) : \mathrm{F}(1+\tan\vartheta/2)]\,d\vartheta\right\}.$$

Thus

$$\arg\frac{dz}{dw}=\arg\frac{(\zeta+1)\sqrt{\zeta-1}}{\sqrt{(\zeta+i)(\zeta-e^{i\alpha})(\zeta-e^{i\beta})}}$$
$$-\frac{3}{2\pi}\int_{3/2\pi}^{2\pi}\tan^{-1}\frac{\mathrm{F}(-\tan\vartheta/2)}{\mathrm{F}(1+\tan\vartheta/2)}\,\mathrm{I}\left(i\frac{e^{i\vartheta}+\zeta}{e^{i\vartheta}-\zeta}\right)d\vartheta+\mathrm{M}. \quad (19)$$

and, therefore,

$$\frac{dz}{dw}=\frac{\mathrm{C}}{k}\frac{(\zeta+1)\sqrt{\zeta-1}}{\sqrt{(\zeta+i)(\zeta-e^{i\alpha})(\zeta-e^{i\beta})}}\cdot$$
$$\cdot\exp\left\{\frac{-3i}{2\pi}\int_{3/2\pi}^{2\pi}\tan^{-1}\frac{\mathrm{F}(-\tan\vartheta/2)}{\mathrm{F}(1+\tan\vartheta/2)}\cdot\frac{e^{i\vartheta}+\zeta}{e^{i\vartheta}-\zeta}\right\}d\vartheta.$$

Here α and β are the above-mentioned parameters and C is the complex constant, of which the argument M depends on α and β, and the modulus is a parameter. (The arbitrary real factor mentioned in §4 is taken in the form $\frac{|\mathrm{C}|}{k}$ so that z should not depend on the percolation coefficient k.)

The formula (16 *a*) gives us

$$\frac{dw}{d\zeta}=k\frac{d}{d\zeta}\left[\mathrm{F}\left(\frac{1-\zeta}{(1-i)-(1+i)\zeta}\right):\mathrm{F}\left(\frac{-i(1+\zeta)}{(1-i)-(1+i)\zeta}\right)\right].$$

Hence

$$h_1=-\frac{1}{i}\int_0^{(w')_{\mathrm{up}}}\frac{dz}{dw}\,dw=\frac{\mathrm{C}i}{k}\int_1^{e^{i\alpha}}\left\{\frac{(\zeta+1)\sqrt{\zeta-1}}{\sqrt{(\zeta+i)(\zeta-e^{i\alpha})(\zeta-e^{i\beta})}}\right.$$
$$\times\exp\left[\frac{-3i}{2n}\int_{3/2\pi}^{2n}\tan^{-1}\frac{\mathrm{F}(-\tan\vartheta/2)}{\mathrm{F}(1+\tan\vartheta/2)}\frac{e^{i\vartheta}+\zeta}{e^{i\vartheta}-\zeta}d\vartheta\right].k.$$
$$\left.\times\frac{d}{d\zeta}\frac{\mathrm{F}\left(\frac{1-\zeta}{(1-i)-(1+i)\zeta}\right)}{\mathrm{F}\left(\frac{-i(1+\zeta)}{(1-i)-(1+i)\zeta}\right)}\right\}d\zeta,$$

or, taking into consideration that on the upper-water boundary the real part of $\frac{dz}{dw}$ and the imaginary part

of dw are equal to zero, and the imaginary part of dz does not change its sign, we have

$$h_1=|C|\int_1^{e^{i\alpha}}\left|\frac{(\zeta+1)\sqrt{\zeta-1}}{\sqrt{(\zeta+i)(\zeta-e^{i\gamma})(\zeta-e^{i\vartheta})}}\right|\cdot$$

$$\times\exp\left\{\frac{3}{2\pi}\int_{3\pi/2}^{2\pi}\tan^{-1}\frac{F(-\tan\vartheta/2)}{F(1+\tan\vartheta/2)}\,I\left(\frac{e^{i\vartheta}+\zeta}{e^{i\vartheta}-\zeta}\right)d\vartheta\right\}\cdot$$

$$\times\left|\frac{d}{d\zeta}\frac{F\left(\frac{1-\zeta}{(1-i)-(1+i)\zeta}\right)}{F\left(\frac{-i(1+\zeta)}{(1-i)+(1+i)\zeta}\right)}\right|d\zeta.$$

Finally, the substitution of $\xi=e^{i\tau}$ gives

$$h_1=|C|\int_0^{\alpha}\left|\frac{\cos\frac{\tau}{2}\sqrt{\sin\frac{\tau}{2}}}{\sqrt{\sin\left(\frac{\tau}{2}-\frac{3\pi}{4}\right)\sin\frac{\tau-\alpha}{2}\sin\frac{\tau-\beta}{2}}}\right|$$

$$\times\exp\left\{\frac{3}{2\pi}\int_{3\pi/2}^{2\pi}\tan^{-1}\frac{F(-\tan\vartheta/2)}{F(1+\tan\vartheta/2)}\cot\frac{\tau-\vartheta}{2}\,d\vartheta\right\}$$

$$\times\frac{d}{d\tau}\frac{F\left(\frac{\tan\vartheta/2}{1+\tan\vartheta/2}\right)}{F\left(\frac{1}{1+\tan\vartheta/2}\right)}d\tau.$$

In the same way it is possible to show that l and h_2 are expressed by the integrals of the same function taken between the limits α to β and β to π respectively.

Thus we have three equations, establishing the dependence between the magnitudes which are given—h_1, l, and h_2—and the parameters of the solution α, β, and $|C|$; and the values of the parameters α and β being chosen, the equations (16) and (20) give us the final solution of the problem.

LXXIII. *On the Steady Motion of Ground-Water through a Wide Prismatic Dam.* *By* B. Davison †.

Introduction.

IN another paper, "On the Steady Two-Dimensional Motion of Ground-Water with a Free Surface," we have given a general solution of the problem of determination of the steady two-dimensional ground-water motion through a uniform permeable ground bounded by plane surfaces. It was also shown that any solution of the problem brought up in the above paper can be obtained by the method therein given.

The aim of the present article is to show that the solution obtained has exactly as many parameters as are necessary to satisfy all the conditions of the problem. This proof is here carried out only for a particular class of regions, viz., those satisfying the conditions specified in § 1. These conditions are not essential to the solution of the problem, and were introduced by the author only for the purpose of simplifying the reasoning. If required the reader may disregard any of them, provided he makes the necessary alterations in the reasoning.

The author's previous paper makes it easy to see that it is only in the determination of the region $w=v_x-iv_y$ by the lines of which the arcs compose the boundary of this region, that more than one way of carrying out the solution is possible. That is, although the lines are given the arcs have to be chosen. Therefore, in order to find the number of parameters on which the solution depends we have to examine certain properties of the w-region, which is done in § 3, and which in its turn necessitates an examination in § 2 of some properties of the complex potential $f=\phi+i\psi$. In § 4 a question of practical importance is discussed, that concerning the possibility of ground-water motion through an earth dam without its oozing out, *i. e.*, without any seepage areas being produced. On the basis of all the preceding the question of determination of the number of parameters is discussed in § 5.

The notation adopted in the present paper is the same that was used in the above-mentioned paper "On Two-

† Communicated by Prof. N. S. Koshliakov.

Dimensional Motion." The x-axis we shall direct towards the tail-water and the y-axis vertically upward. All the contours are supposed to be traversed counter-clockwise.

§ 1. *The Region of Motion.*

We shall suppose that the region of motion is determined by the following conditions :—

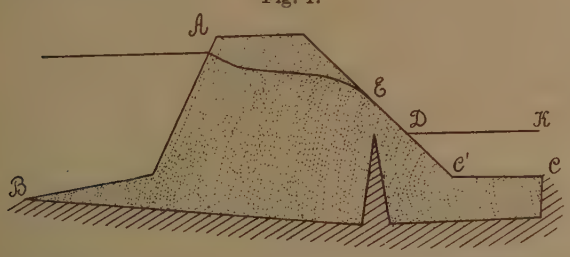
1. There are only two water-basins on the boundary of the region, viz., upper-water and tail-water.

2. There is only one interval of seepage, and it is situated above the tail-water †. We shall regard it as being composed of one rectilinear portion.

There is also only one free surface.

3. There are no hydro-technical structures inside the region of motion, and therefore the region of motion is singly connected.

Fig. 1.



An example of a dam in question, the region AB CC′ DEA being the region of motion.

4. There is no evaporation of water from the free surface as well as no income of water from without upon the free surface or the interval of seepage.

5. The region of motion is finite.

6. If we travel along the boundary counter-clockwise, then along the upper-water y does not increase (*i. e.*, on individual portions—it may either decrease or remain constant) ; along the impervious base x does not decrease (it either increases or remains constant) ; along the tail-water boundary y does not decrease (it increases or remains constant); and along the interval of seepage y increases.

† It is not difficult to show that no interval of seepage is possible above the upper-water.

In fig. 1 is given an example of a permeable dam satisfying the above enumerated conditions.

§ 2. *The Region of the Complex Potential.*

Let us take an auxiliary complex variable ζ and let us consider the function $f(\zeta)$ giving the conformal representation of the circle $|\zeta| \leqslant 1$ on the region of the complex potential $f=\phi+i\psi$ †. On the impervious base the stream function $\psi=\mathrm{I}(f)$ is constant. In f-plane the corresponding portion of the boundary will be represented by a straight line segment. But in the ζ-plane we shall have an arc of the circumference $|\zeta|=1$ traversed always in one direction. Therefore, in accordance with Schwarz's principle of symmetry, at any point on the above arc (with the exception, of course, of the end points of this arc) $f(\zeta)$ is analytic, independently of the fact whether the point in z-plane, corresponding to the point in question, lies on a rectilinear portion of the impervious base. or whether it is the common end of two such portions ‡.

Hence $f(\zeta)$ may be analytically continued from the region $|\zeta|<1$ to the whole of the region $|\zeta|>1$, and then the velocity potential $\phi=\mathrm{R}(f)$ must have the same value at conjugate points (ζ and $1/\zeta$). But we have just seen that ϕ is a regular harmonic function in the whole of the region composed of $|\zeta|>1$, $|\zeta|<1$ and the arc in question. Therefore ϕ cannot reach its greatest or smallest value (maximum maximorum and minimum minimorum) on this arc, except, perhaps, at the end points.

In accordance with assumption 4, the free surface is a stream-line, and therefore the above reasonings apply to this part of the boundary as well. Hence on the free surface (except, perhaps, at its end points), ϕ also cannot have its greatest or smallest value.

Comparing the boundary conditions on other portions of the contour we see that on the tail-water the values

† We shall define ϕ so that along a stream-line it would increase in the direction of motion. Condition 3 being satisfied, the region $f=\phi+i\psi$ is singly connected.

‡ If the corresponding straight line segment $\psi=$const. in the plane f is not all the time traversed in one and the same direction, that is, if it has maxima and minima, then at the corresponding point $f'(\zeta)=0$, so that $\zeta(f)$ will not be analytic and will have an algebraical branch-point, but $f(\zeta)$ will be analytic in any case.

of the velocity potential are greater than those on the upper-water boundary, and on the interval of seepage. Hence it is just on the tail-water boundary that the velocity potential reaches its maximum value.

In virtue of the boundary conditions, ϕ is constant along lines forming the boundary of water-basins. Therefore the derivative of the velocity potential taken on an inner normal $\frac{\partial \phi}{\partial n}$ is negative along the whole boundary of the tail-water. Hence, bearing in mind that

$$\frac{\partial \phi}{\partial n} = -\frac{\partial \psi}{\partial s},$$

we see that on the tail-water boundary the stream-function steadily increases.

In accordance with assumption 4, there is no income of water from without upon the interval of seepage. This means that on any portion of the interval of seepage, beginning at the upper point of this interval and ending on it at any arbitrary point, the amount of water oozing out of the soil is not smaller than the amount of water penetrating back into the soil. Therefore the value of the stream-function at the upper point of the interval of seepage is not less than that at any other point of this interval. But the upper point of the interval of seepage belongs also to the free surface, and on the latter ψ is constant. Hence the value of the stream function ψ on the free surface is not less than its values at any point of the interval of seepage, and so much more so than those on the impervious base.

On the upper-water boundary ϕ is constant, and therefore ψ cannot reach its greatest value on the corresponding part of the boundary except, perhaps, at its end points. (The proof of this is exactly analogous to the proof given in the beginning of this paragraph.) Thus we come to the conclusion that the greatest value of the stream-function ψ is reached on the free surface. Therefore, since $\frac{\partial \psi}{\partial n} = \frac{\partial \phi}{\partial s}$, ϕ decreases along the free surface. On the seepage surface the pressure p is constant, y increases and hence, as can be seen from the formula $\phi = -k\left(\frac{p}{\rho g} + y\right)$, the velocity potential ϕ will be

decreasing along this surface. As has already been mentioned, the velocity potential will be decreasing along the free surface. Hence, in the upper-water boundary ϕ is less than on the seepage surface. Bearing in mind all that has been said about ϕ we come to the conclusion that it is just on the upper-water boundary that this function reaches its smallest value in the region. It follows from what we have proved above that in the passage of this boundary the stream-function ψ decreases steadily.

In order definitely to ascertain the outline of the f-region, it remains only to find out whether the values of the stream-function on the seepage surface can be smaller than those on the impervious base. Traversing the seepage surface on the z-plane and moving from the tail-water towards the free surface, we have the region of motion on our left, and this will be true for the f-region as well. But since ϕ in this case, as we have seen, steadily decreases, the formula $\frac{\partial\phi}{\partial s} = \frac{\partial\psi}{\partial n}$ shows that if we leave the boundary and travel in the region along a line "ϕ=const.", ψ will be decreasing. Hence it follows that ψ cannot reach its smallest value on the seepage surface. The ends of the seepage surface boundary belong to other portions of the contour which have already been examined and about which it has already been proved that ψ cannot reach there its smallest value. Thus ψ has its smallest value on the impervious base. Therefore (remembering that on the impervious base ψ is constant) in traversing this base from the upper-water towards the tail-water, ϕ must increase steadily.

Comparing all the conclusions obtained above we see that the region of the complex potential will have the outline in fig. 2, the shape of the curve DE, and the lengths of the straight lines AB, BC, CD, and EA depend on the configuration of the dam. The above proofs also show that the f-region boundary does not intersect itself anywhere, and since f is analytic within the region $|\zeta| < 1$, we may also conclude that the inverse function $\zeta(f)$ is single valued. It is also evident that the f-region is singly-connected.

Let us, moreover, examine a complex quantity

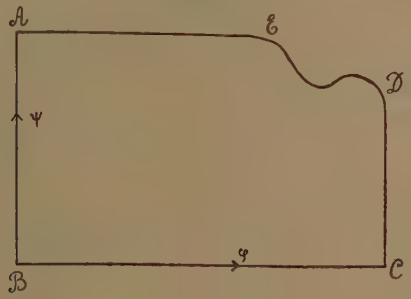
$$P=\rho g\left(iz-\frac{f}{k}\right)=p+i\tilde{p}, \quad . \quad . \quad . \quad . \quad (1)$$

where p is the pressure and $\tilde{p}$ is its conjugate harmonic function. Along the water-basin boundaries ϕ is constant and the variation of y in virtue of the assumption 6 is monotonic; hence the variation of

$$p=-\rho g\left(y+\frac{\phi}{\kappa}\right)$$

is also monotonic. Therefore p has its smallest value either on the impervious base or on the free surface

Fig. 2.



and on seepage surface. Traversing the boundary counter-clockwise, x increases along the base (or, more exactly, does not decrease) in accordance with assumption 6 and ψ, as we have seen, is constant. Hence

$$\tilde{p}=\rho g\left(x-\frac{\psi}{\kappa}\right)$$

also does not decrease. Therefore, if we leave the boundary and move along a line "$\tilde{p}=$const," within the region, p will be decreasing.

If, on the other hand, we are on such a portion of the impervious base for which $\tilde{p}$ is const., then before leaving the boundary along the line "$\tilde{p}=$const." into the region we must reach the end of this portion. Let us remember that in traversing the boundary counter-clockwise,

$\tilde{p}$ does not decrease along the impervious base. Therefore, in order to leave the boundary and to move into the region along the line $\tilde{p}$ = const., we must move along the above-mentioned portion in the direction of the decreasing values of p. When we at last leave the boundary we shall have even smaller values of p than on the whole of this portion. If the interval where $\tilde{p}$ = const. abuts on one of the ends of the impervious base and if moving in the direction of p decreasing, we approach this end, then it may happen that we shall not succeed in leaving the boundary and entering the region moving along the line $\tilde{p}$ = const. But in this case we have already a point belonging also to another portion of the boundary, namely, the above-mentioned end point of the base at which the value of p is smaller than at the other points of our portion.

Thus p cannot have its smallest value (minimum minimorum) at any point of the impervious base —with the exception maybe of its ends. But these ends belong to the boundaries of the water-basins. The variation of p on these boundaries has been examined above, and we have seen that on the ends of the impervious base p also cannot have its smallest value.

Comparing all the above we see that p can have its smallest value only either on the free surface or on the seepage surface. But in virtue of the boundary conditions p is constant on the free surface and on the seepage surface, and has the same value on both these positions. Therefore, $\frac{\partial p}{\partial n}$ (the derivative on the inner normal) is positive at any point on these two portions, and hence $\frac{\partial \tilde{p}}{\partial s} = -\frac{\partial p}{\partial n}$ is negative and $\tilde{p}$ steadily decreases.

Since $\tilde{p} = \rho g\left(x - \frac{\psi}{\kappa}\right)$ and ψ is constant on the free surface, x also steadily decreases on the free surface, when the boundary is traversed counter-clockwise.

§3. *Directions of the Velocities.*

The conclusions arrived at in the preceding paragraph permit us to ascertain within what limits the direction of the velocity vector can vary. First let us examine how the direction of the velocity vector varies on the

boundary. The angle determining the direction of velocity is

$$\tan^{-1}\frac{v_y}{v_x} = -\tan^{-1}\frac{-v_y}{v_x} = -\arg w = -\arg\frac{df}{dz} = \arg\frac{dz}{df}.$$

To find this angle let us examine the boundary values of $\arg dz$ and $\arg df$, by which symbols we denote $\arg\left(i\zeta\frac{dz}{d\zeta}\right)$ and $\arg\left(i\zeta\frac{df}{dz}\right)$, where ζ is an auxiliary complex variable introduced in § 2.

Since the argument is many-valued, we shall choose it in such a way that we should have its principal values on the first portion of the impervious base, when we traverse the boundary for the first time.

Then making use of assumption 6 and the conclusions obtained in § 2 (see fig. 2) we have

1. On the upper-water boundary :

$$-\pi \leqslant \arg dz \leqslant 0\ ;\ \arg df = -\frac{\pi}{2};\ \text{hence} -\frac{\pi}{2} \leqslant \arg\frac{df}{dz} \leqslant \frac{\pi}{2}.$$

2. On the impervious base :

$$-\frac{\pi}{2} \leqslant \arg dz \leqslant \frac{\pi}{2};\ \arg df = 0\ ;\quad \text{hence}\ -\frac{\pi}{2} \leqslant \arg\frac{df}{dz} \leqslant \frac{\pi}{2}.$$

3. On the tail-water boundary :

$$0 \leqslant \arg dz \leqslant \pi\ ;\ \arg df = \frac{\pi}{2};\quad \text{hence}\ -\frac{\pi}{2} \leqslant \arg\frac{df}{dz} \leqslant \frac{\pi}{2}.$$

4. On the seepage surface :

$$0 < \arg dz < \pi\ ;\ \frac{\pi}{2} < \arg df < \frac{3}{2}\pi;\ \text{hence}\ -\frac{\pi}{2} < \arg\frac{df}{dz} < \frac{3\pi}{2}.$$

5. On the free surface :

$$\frac{\pi}{2} < \arg dz < \pi\ ;\ \arg df = \pi\ ;\quad \text{hence}\ 0 < \arg\frac{df}{dz} < \frac{\pi}{2}.$$

. . . (2)

Since, for the seepage surface, both $\arg dz$ and $\arg df$ are connected only by inequalities, we can expect that the above obtained limits within which $\arg\frac{df}{dz}$ varies on the seepage surface are too broad, and we must try to make them more precise. Let us use for this purpose

the function P introduced in § 2, and let us show that on the seepage surface the horizontal component of the velocity $v_x \left(=\mathrm{R}(w)=\mathrm{R}\left(\frac{df}{dz}\right)\right)$ is not negative.

In fact, in virtue of the formula (1), $f=k\left(iz-\frac{\mathrm{P}}{\rho g}\right)$, and therefore

$$v_x=\mathrm{R}\left(\frac{df}{dz}\right)=\mathrm{R}\left(ki-\frac{k}{\rho g}\frac{d\mathrm{P}}{dz}\right)=-\frac{k}{\rho g}\mathrm{R}\left(\frac{d\mathrm{P}}{dz}\right)$$

$$=-\frac{k}{\rho g}\left|\frac{d\mathrm{P}}{dz}\right|^2\mathrm{R}\left(\frac{dz}{d\mathrm{P}}\right)=-\frac{k}{\rho g}\left|\frac{d\mathrm{P}}{dz}\right|^2\frac{\partial y}{\partial \tilde{p}}. \quad (3)$$

The partial derivative $\frac{\partial y}{\partial \tilde{p}}$ is a derivative taken along the line $p=$const.; and the interval of seepage is just such a line. When we move along the interval of seepage towards the free surface, then, as we have seen, $\tilde{p}$ decreases and y, in virtue of condition 6, must increase. Therefore $\frac{\partial y}{\partial \tilde{p}}$ is negative at all inner points of the seepage surface, and hence $v_x=\mathrm{R}\left(\frac{df}{dz}\right)$ must be positive. But in this case $\arg\frac{df}{dz}$ can be written in the form

$$\arg\frac{df}{dz}=\alpha+2\pi n, \qquad \alpha\left|\leqslant\frac{\pi}{2}\right.,$$

where n is an integer. Comparing this with the formula (2) we see that n can be equal only to zero, *i. e.*, on the interval of seepage we have the inequality

$$\left|\arg\frac{df}{dz}\right|\leqslant\frac{\pi}{2}.$$

Thus we see that the following inequality

$$-\frac{\pi}{2}\leqslant\arg w\leqslant\frac{\pi}{2}, \quad (\text{hence } \mathrm{R}(w)\geqslant 0), \quad (4)$$

holds good for the whole of the boundary. In order to prove that this inequality is satisfied in the whole of the region, let us show, first, that the boundaries of z-region and f-region are rectifiable. Let us, *e. g.*, take the boundary of f-region. The portions of the

boundary corresponding to the free surface, water-basins, and the impervious base, are straight line segments of finite length, each of which is traversed only once when we travel once around the boundary (see § 2). Hence their total length is finite. As to the seepage surface ϕ is monotonic there, and $\psi=k\left(x-\frac{\tilde{p}}{\rho g}\right)$ has a limited total fluctuation, since both x and $\tilde{p}$ on the seepage surface vary monotonically and are bounded on this surface.

Therefore the length of the portion of f-region corresponding to the seepage surface is also finite.

In an exactly similar way we can treat the z-region boundary. On the other hand, we have the following mathematical lemma :—

Let Z be an analytic function of a complex variable given on the circle $|\zeta|<1$, and let the region of its values be finite, singly-connected, and bounded by a rectifiable contour and, moreover, let the inverse function $\zeta(Z)$ be single-valued. Then if the boundary values of $\arg\frac{dZ}{d\zeta}$ are summable (L), the analytic function $\log\frac{dZ}{d\zeta}$ can be represented by Poisson's integral †.

All the conditions of this lemma are satisfied both for z and for f (and $\arg\frac{dz}{d\zeta}$ and $\arg\frac{df}{d\zeta}$ are not only summable (L) but are even bounded). Therefore $\log\frac{dz}{d\zeta}$ and $\log\frac{df}{d\zeta}$ can be represented by Poisson's integral, and the same holds true for their difference, and the more so for the imaginary part of this difference, *i. e.*, for

$$\mathrm{I}\left(\log\frac{df}{d\zeta}-\log\frac{dz}{d\zeta}\right)=\mathrm{I}\left(\log\frac{df}{dz}\right)=\arg w.$$

Therefore, the inequality (4) proved above for the boundary must be satisfied in the whole of the region, and hence the horizontal component of the velocity cannot be negative anywhere.

† See Smirnov, "Sur les formules de Cauchy et de Green," *Bulletin de l'Academie des Sciences de l'U. R. S. S.* 1932, p. 337 ff. . . . . The above result is given on pp. 352–53, as it can be seen from his formula (19) (p. 348), and from the definition of the regions of α-type (p. 341).

Let us, by the way, point out the following circumstance.

In the author's previous paper the question was brought up as to whether some solutions would not be lost in the determination of $\arg \frac{dz}{d\zeta}$ in the region by its boundary values by means of Poisson's integral. Having proved that $\log \frac{dz}{d\zeta}$ can be represented by Poisson's integral, we have removed these doubts for the case where the region satisfies the conditions enumerated in § 1.

§ 4. *On the Seepage Surface.*

The above results permit us to draw some interesting and useful conclusions concerning the general character of the motion. For instance, it appears that a seepage surface is unavoidable if the upper portion of the tail-water boundary forms a right or an acute angle C′DK with the horizontal surface of the tail-water (see fig. 1). The motion of the ground-waters without a seepage surface is possible only when the above-mentioned angle C′DK is an obtuse angle. Let us suppose that the angle C′DK is a right angle or an acute angle. Since the velocity vector at the water-basin boundary is perpendicular to this boundary, and since the horizontal component of the velocity is not negative, the vertical component also is not negative on this portion of the boundary. Then $\mathrm{I}(w) = -v_y$ is not positive. On the free surface, on the contrary, $0 < \arg w < \frac{\pi}{2}$ (see form. (2)), and hence $\mathrm{I}(w)$ is positive. Moreover, in another paper the author has demonstrated that on the free surface $\mathrm{I}(w)$ is bounded. Let us suppose now that there is no seepage surface and that the free surface abuts directly on the tail-water boundary. Then, since the boundary of the region of a complex variable is always continuous, $\mathrm{I}(w)$ must equal zero at the point where the free surface abuts on the tail-water boundary.

The equation of the free surface in the plane w, $\mathrm{I}\left(\frac{k+iw}{w}\right) = 0$ (where k is a real positive number), shows

that w itself must equal zero at the above-mentioned point. When we travel in z-plane along the boundary in a counter-clockwise direction, $I(w)$ will at the above point change from negative, or zero values, to positive values. Therefore anywhere in the vicinity of the point in question intervals can be found along which $I(w)$ is increasing. Hence either at all points lying sufficiently close to the point in question $I(w)$ increases monotonically, or anywhere in the vicinity of this point $I(w)$, has an infinite number of maxima and minima. Let us examine the first supposition; the impossibility of the second will be shown in § 5 (see footnote, p. 921). $R(w)$ must be positive at any point within the region, and in the point under consideration, *i. e.*, where the free surface abuts on the tail-water boundary, $R(w)$ is equal to zero (see above). Thus we come to the conclusion that if we traverse this point in the w-plane travelling from the tail-water boundary towards the free surface then we shall have the region on the right-hand side. This contradicts the well-known property of analytic functions, viz., that the direction in which the boundary is being traversed is not changed in conformal representation. The cause of this contradiction can only lie in the supposition of the absence of a seepage surface. Thus we have ascertained that a seepage surface is unavoidable when the conditions are as outlined above.

If, on the contrary, the angle C′DK is obtuse, then $I(w)$ is positive both on the free surface and on the upper portion of the tail-water boundary, and therefore the corresponding lines in the w-plane intersect at two points, viz., at the point $w=0$ and at another point having a positive value $R(w)$.

Thus when the angle C′DK is obtuse, one of three cases is possible, viz. :—

(*a*) A seepage surface exists.

(*b*) The free surface abuts directly on the tail-water boundary, which at the common point of these portions has the above-mentioned positive value of $R(w)$.

(*c*) The free surface abuts directly on the tail-water boundary, and at the common point of these portions $w=0$.

In the simplest example of percolation through a rectangular massive on an inclined base, we can show

that when the angle C′DK is obtuse, then depending on the relation of the upper- and tail-water levels (and some other linear dimensions in case of a more complicated configuration of the dam), any of the above enumerated possible cases can be realized.

§ 5. *On the Number of Parameters.*

Now we may pass to the determination of the number of parameters. The proofs we shall give here apply only to the case where there is a seepage surface. For other cases we shall limit ourselves to giving the corresponding results.

Let n_1 be the number of rectilinear portions of which the upper-water boundary is composed, n_2 the number of rectilinear portions of the impervious base, n_3 those of the tail-water boundary, and let $n=n_1+n_2+n_3$. The configuration of the dam is determined by the lengths of these n portions and by the angles between them.

The magnitudes of these angles have already been made use of in the determination of the lines composing the boundary of w-region, and they can give us no other conditions.

Hence the lengths of these n-portions of the boundary are the only parameters of the problem.

The boundary of the w-region consists of $n+2$ portions, viz.: n_1 portions corresponding to the upper-water boundary, n_2 portions corresponding to the impervious base, n_3 portions corresponding to the tail-water boundary, one portion corresponding to the seepage surface and one portion corresponding to the free surface. The first $n=n_1+n_2+n_3$ portions of the boundary of w-region are, as the boundary conditions show, segments of straight lines passing through the point $w=0$. Therefore, if we assume

$$\arg dw=\alpha+m\pi, \qquad |\alpha| \leqslant \frac{\pi}{2}, \quad (5)$$

where m is an integer, and $\arg dw$ denotes $\arg\left(i\zeta\frac{dw}{d\zeta}\right)$, then along these portions $\alpha=\arg w$ $\left(\text{we have to remember that } |\arg w| \leqslant \frac{\pi}{2}\right)$. The point where two adjacent

portions join will be either $w=0$ or $w=\infty$ with the exception of the case when in the plane w the straight lines, on which these portions lie, coincide.

In the case when the above straight lines coincide the position of the point separating one portion from another is a parameter. Let us now examine the point where two portions join and for which $w=0$.

The boundary of w-region forms an angle at this point. Let us denote its magnitude by β. Since $\mathrm{R}(w) \geqslant 0$, this angle is convex, and when we pass its vertex (in moving around the boundary counter-clockwise) $\arg dw$ increases by $\pi-\beta$, whilst $\arg w$ decreases by β. Hence the integer m in formula (5) increases by 1. The same is true of the points, where the portions join, and for which $w=\infty$. If we have $w=0$ for both end points of a boundary portion of the region of motion, then the boundary values of w must have a maximum between these points. The magnitude of this maximum is a parameter. On passing this maximum,

$$\arg dw = \arg\left(i\zeta \frac{dw}{d\zeta}\right)$$

is decreased by π. $\alpha=\arg w$ is not changed, and therefore the integer m in formula (5) decreases by unity. The same conclusions hold good whenever inside of one of the n-portions in question $|w|$ has a minimum or a maximum, and they will also be true for any maximum and minimum of $\mathrm{R}(w)$ on the seepage surface and any minimum and maximum of $|w|$ on the free surface.

On passing from the upper portion of the upper-water boundary to the upper portion of the tail-water boundary moving in a counter-clockwise sense, the integer m in formula (5) will thus increase by

$$(n-1)-j_1,$$

where j_1 is the number of parameters acquired on the above-mentioned path. Let us now examine the point where the tail-water boundary joins the seepage surface. If in this point $w=\infty$, then, in the same way as above, we can show that the integer m in formula (5) increases by unity when we pass this point. Let us now examine the case when at this point w is not infinity. The seepage surface is represented on the plane w by a straight line segment passing through the point $w=ik$, which follows

from the boundary conditions given in another paper by the present writer †. Therefore, on the seepage surface

$$\arg(w-ik)=\tan^{-1}\left\{\frac{\mathrm{I}(w)-k}{\mathrm{R}(w)}\right\}$$

has a constant value. We have to remember that along the upper portion of the tail-water boundary

$$\arg w=\tan^{-1}\frac{\mathrm{I}(w)}{\mathrm{R}(w)}$$

also has a constant value. Let us take a point common to both lines. At this point $\mathrm{R}(w)$ is positive, k is a positive constant and therefore

$$\tan^{-1}\frac{\mathrm{I}(w)}{\mathrm{R}(w)}>\tan^{-1}\frac{\mathrm{I}(w)-k}{\mathrm{R}(w)},$$

that is, at the point in question

$$\arg w>\arg(w-ik).$$

Hence the value of $\arg w$ on the upper portion of the tail-water boundary is greater than that of $\arg(w-ik)$ on the seepage surface. Therefore, if the angle formed by the portions of the boundary at the point under consideration in the w-plane is convex $\mathrm{R}(w)$ has a maximum or a minimum at this point. It is not difficult to show this geometrically, if we remember that in passing from the line corresponding to the tail-water boundary to that corresponding to the seepage surface we have the region on the left-hand side. In passing a minimum or a maximum of $\mathrm{R}(w)$ the integer m in formula (5) is increased by unity; this can be proved in exactly the same way as for the points where $w=0$ and where $w=\infty$.

If, on the other hand, the angle in the w-plane, at the point where the tail-water boundary joins the seepage

† See Davison, "On the Steady Two-dimensional Motion of Ground-Water with the Free Surface."

For the seepage surface the equations (10) of the above-mentioned paper must be written in the form

$$\mathrm{I}(kz+if)=\text{const.}; \quad \mathrm{I}(m_{\mu+1}z)=\text{const.}$$

and hence the equation (13) of the same paper will be $\mathrm{I}(k+iw)=0$. This is the equation of a straight line in the w-plane. Putting $w=ik$ we shall have an identity, which shows that this line passes through the point $w=ik$.

surface, is concave, then we can show again, applying the above reasonings, that in this case the variation of $R(w)$ is monotonic and the integer m in formula (5) does not undergo any changes. Hence, in passing from the tail-water boundary on the seepage surface, m increases by unity, unless the angle at the dividing point is concave.

In exactly the same way we may examine the variation of the magnitudes in the formula (5) at the end points of the free surface. Therefore, we give only the results. If at the point where the seepage surface joins the free surface $I(w) > \frac{k}{2}$, we have either a concave angle in the w-plane, in which case m does not change, or m increases by unity, in which case the angle is convex. But if $I(w) < \frac{k}{2}$, then either the angle is concave and m decreases by unity, or the angle is convex and m does not change At the limit, *i. e.*, if $I(w) = \frac{k}{2}$ at the point where the two portions join, we shall have the same conditions as when $I(w) > \frac{k}{2}$, if at the adjacent points of the free surface $I(w) > \frac{k}{2}$ and the same conditions as when $I(w) < \frac{k}{2}$ in the opposite case.

On the other hand, at the point where the free surface abuts on the upper-water boundary, if at this point $I(w) > \frac{k}{2}$, we have either a concave angle and m decreases by unity, or the angle is convex and m does not change. But if at this point $I(w) < \frac{k}{2}$, then we either have a concave angle and m does not change, or m increases by unity and the angle is convex.

Let us now make an observation, which will also explain why the above given result depended every time on the sign of the difference $I(w) - \frac{k}{2}$. At the point

of the portion of the w-region boundary corresponding to the free surface, the tangent to this portion is vertical if $w=k^{\frac{1+i}{2}}$ for the point in question. Therefore, when we pass this point travelling along the portion of the boundary of w-region corresponding to the free surface, arg w passes through $\pi(p_1+\frac{1}{2})$ (where p_1 is an integer), increasing if $\mathrm{I}(w)$ increases and decreasing in the opposite case.

But since in formula (5) $|\alpha|$ is not greater than $\frac{\pi}{2}$, the integer m must change in passing this point, increasing by unity if $\mathrm{I}(w)$ increases and decreasing by unity in the opposite case.

Combining the different suppositions concerning the sign of the difference $\mathrm{I}(w)-\frac{k}{2}$ on both ends of the free surface we shall obtain only four possible cases. Let us analyse one of them in detail. Let us suppose, for instance, that at the point where the seepage surface joins the free surface $\mathrm{I}(w)>\frac{k}{2}$, and at the point where the free surface abuts on the upper-water boundary $\mathrm{I}(w)<\frac{k}{2}$. Let us find for this case the increase of the integer m when we travel once round the boundary. Let us denote the number of maxima and minima of $\mathrm{R}(w)$ on the seepage surface by j_2 and the number of maxima and minima of $|w|$ on the free surface by j_3. In passing from the upper point of the upper-water boundary to the upper point of the tail-water boundary and moving along the boundaries of water-basins and of the impervious base, m will increase by $(n-1)-j_1$; on the seepage surface m decreases by j_2, on the free surface m decreases by j_3+1 (we have to remember that in virtue of the above supposition concerning the end points of the free surface, we shall pass the point $w=k^{\frac{1+i}{2}}$, travelling in the direction of $\mathrm{J}(w)$ decreasing, one time more than when we move in the direction of $\mathrm{J}(w)$ increasing). Moreover, we have three more points—the ends of the seepage surface and of the free

surface,—and at each of these points we have either m increasing by unity or concave angle in the w-plane. If we denote the number of such angles by j^*, then the increase of m when we travel once around the boundary will be

$$[(n-1)-j_1]-j_2-(j_3+1)+(3-j^*)$$
$$=n+1-(j_1+j_2+j_3)-j^*=n+1-j-j^*, \quad (6)$$

where $j=j_1+j_2+j_3$ is the total number of the points on the boundary of the w-region, which we may regard as parameters.

The same value of the total increment of the integer m in formula (5) will be obtained in case of the three other suppositions concerning the sign of the difference $I(w)-\frac{k}{2}$ at the ends of the free surface. On returning to the starting-point α returns to its initial value, and hence the increment of $\arg dw=\arg\left(i\zeta\frac{dw}{d\zeta}\right)$ is equal to $\pi(n+1-j-j^*)$. But $\arg\zeta$ increases by 2π when we travel once round the boundary, and hence $\arg\frac{dw}{d\zeta}$ increases by $\frac{n-1-j-j^*}{2}\cdot 2\pi$, and, in accordance with the well-known Cauchy's theorem, $\frac{dw}{d\zeta}$ has $\frac{n-1-j-j^*}{2}$ zeros within the region †.

Therefore $\zeta(w)$ has $\frac{n-1-j-j^*}{2}$ branch points, and these points will also be the branch points of $z(w)$ and $f(w)$. Some of these branch points can coincide. The position of any of these branch points in the w-plane is determined by a complex number, *i. e.*, by two real numbers, and hence these branch points are equivalent to $(n-1-j-j^*)$ parameters.

Let us now examine the concave angle. If we produce one of its sides to a very small length ϵ beyond its vertex, and if we return then to this vertex, we shall acquire a new parameter and the concave angle will be replaced

† If there was no seepage surface, we should have for the number of zeros a similar formula, which would differ only in details; and since $w(\zeta)$ is analytic in the region and the number of zeros cannot be negative, then j and the more so the number of maxima of $R(w)$ on the free surface in the vicinity of the point $w=0$, are finite. Thus the statement that we made use of in § 4 is proved.

by a convex angle. Therefore, to every one of the concave angles we can give one more parameter. Then the total number of parameters determining the w-region will be

$$j+(n-1-j-j^*)+j^*=(n-1).$$

In the determination of $\frac{dz}{dw}$ an arbitrary real factor will be introduced, as the author has shown in another paper. Thus the solution depends on n-parameters, that is, on exactly as many parameters as can be determined by the data of the problem.

Examining exactly in the same way the case when the tail-water boundary abuts directly on the free surface and at that point $w=0$, we come to the conclusion that in this case also the number of parameters on which the solution depends is equal to the number of parameters on which depend the data of the problem.

But in the case when the tail-water boundary abuts directly on the free surface and at that point the velocity is not zero (case *b*), the number of parameters on which the solution depends is smaller by unity than the number of portions of which the boundaries of the water-basins and of the impervious base are composed. In this case the outline of the dam and one of the water levels will determine the other level.

LXXIV. *Ring and Disk Sources.* *By* GEORGE GREEN, *D.Sc., Lecturer in Physics in the Applied Physics Department of the University of Glasgow* *.

IN a number of previous papers † the present author has developed a very general method of solution of problems in mathematical physics by means of wave-trains. The method has been illustrated throughout by the discussion of problems in heat conduction. All the problems solved in these papers involve solutions of the fundamental equation of heat conduction

$$\frac{dv}{dt}=\kappa\nabla^2(v). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Communicated by the Author.

† See Phil. Mag. xviii. p. 625 (Oct. 1934).

But the solutions obtained in every case depend upon special solutions fulfilling the condition that v varies with the time t according to the factor e^{ikt}, so that the fundamental solutions employed satisfy the equation

$$(\nabla^2+\lambda^2)v=0\,; \quad \lambda^2=-\frac{ik}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In like manner all problems in the theory of sound involve solutions of the fundamental equation

$$\frac{d^2\phi}{dt^2}=\mathrm{V}^2\nabla^2(\phi)\,; \quad \mathrm{V}=\text{constant}, \quad . \quad . \quad . \quad (3)$$

and if ϕ is proportional to e^{ikt}, the special solutions satisfying this condition must also satisfy equation (2) with λ^2 representing (k^2/V^2). The differential equation (2) arises in all subjects included within the general title of wave-motion. Accordingly it has already been pointed out that the methods employed in investigations in heat conduction can be employed in other subjects, and that the solutions obtained can be transferred by analogy to a large number of different applications. In this connexion the periodic point source heat solution is one of the two fundamental solutions. The ring and disk source solutions form an important group derived from the point source solution, and are of special interest in connexion with certain diffraction problems to be discussed later.

The expression representing a periodic heat source of strength qe^{ikt} at the origin must satisfy equation (2), and must fulfil the additional condition

$$\operatorname*{Lt}_{r=0}\left(-4\pi r^2\mathrm{K}\frac{dv}{dr}\right)=qe^{ikt}. \quad . \quad . \quad . \quad . \quad (4)$$

Both conditions are satisfied by

$$v=\frac{qe^{ikt}}{4\pi\mathrm{K}}\frac{e^{-i\lambda r}}{r}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is the well-known periodic point source solution. In terms of Bessel functions it is

$$v=\frac{qe^{ikt}}{4\pi\mathrm{K}}\sqrt{\frac{2}{\pi}}i\lambda\frac{\mathrm{K}_{\frac{1}{2}}(i\lambda r)}{(i\lambda r)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad (6)$$

In the theory of sound the corresponding solution represents a periodic source of fluid at the origin. Taking qe^{ikt} to be the rate of supply of fluid per unit time from

the origin as source, we express the additional condition to be fulfilled by the periodic source solution by

$$\underset{r=0}{\mathrm{Lt}} \left(4\pi r^2 \rho \frac{d\phi}{dr}\right) = qe^{ikt}, \quad \ldots \quad (7)$$

where ρ represents the density of the fluid and $\frac{d\phi}{dr}$ the outward radial fluid velocity. The required point source solution of strength qe^{ikt} at the origin is then

$$\phi = -\frac{qe^{ikt}}{4\pi\rho} \frac{e^{-i\lambda r}}{r}, \quad \lambda = k/\mathrm{V}, \quad \ldots \quad (8)$$

which can be expressed in terms of the Bessel function $\mathrm{G}_{\frac{1}{2}}$ in the form

$$\phi = \frac{qe^{ikt}}{4\pi\rho} i\sqrt{\frac{2\lambda}{\pi}} \cdot \frac{\mathrm{G}_{\frac{1}{2}}(-\lambda r)}{r^{\frac{1}{2}}} \quad \ldots \quad (9)$$

Between the two Bessel functions introduced above there exists the relation

$$\mathrm{G}_{\frac{1}{2}}(x) = \frac{1}{\sqrt{i}} \mathrm{K}_{\frac{1}{2}}(-ix). \quad \ldots \quad (10)$$

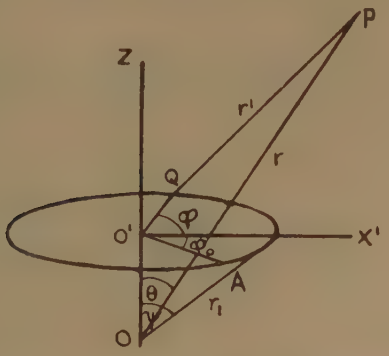
The parallelism between solutions in heat conduction and corresponding solutions in the theory of sound and wave motion illustrated by equations (6) and (9) holds throughout, and need not be pursued further at present. It is sufficient to note that results obtained below for the case of heat conduction have their analogues in other wave-motion subjects.

Periodic Sources.

Periodic Circular Line Source.

Take as a circular line source of strength qe^{ikt} per unit length the circle whose centre O′ is on the axis of z at position $z = r_1 \cos\psi$, and whose plane is perpendicular to axis OZ. Each radius of the circle is then of length $r_1 \sin\psi$ and subtends the angle ψ at the origin as indicated in the accompanying diagram. Let the effect of the circular line source be required at P, any point in space whose position is defined by the coordinates (r, θ). The projection of OP, r, on the plane of the circle gives a reference line which we can denote by O′X′, and the

position of any point Q on the circular line can be defined by the angle ϕ, where $\phi = \angle QO'X'$. An elementary length of the circular line taken at Q. $r_1 \sin\psi d\phi$, is now to be considered as a point source of strength $qr_1 \sin\psi\, d\phi e^{ikt}$, whose distance QP from P is represented



by r'. With $qr_1 \sin\psi d\psi d\phi$ replacing q in (6), and r' replacing r, the effect at P of the complete circular line source is represented by

$$v = \frac{qr_1 \sin\psi e^{ikt}}{4\pi K} \sqrt{\frac{2}{\pi}} i\lambda\, 2 \int_0^{\pi} d\phi \frac{K_{\frac{1}{2}}(i\lambda r')}{(i\lambda r')^{\frac{1}{2}}} . \quad . \quad (11)$$

In this $$r'^2 = r^2 + r_1^2 - 2rr_1 \cos(\mathrm{POQ}), \quad . \quad . \quad . \quad (12)$$

where $$\cos \mathrm{POQ} = \cos\theta\cos\psi + \sin\theta\sin\psi\cos\phi. \quad . \quad (13)$$

According to the addition theorem for Bessel functions we may now write

$$\frac{K_{\frac{1}{2}}(i\lambda r')}{(i\lambda r')^{\frac{1}{2}}} = (2\pi)^{\frac{1}{2}} \sum_0^{\infty} (n+\tfrac{1}{2}) \frac{p_{n+\frac{1}{2}}\{i\lambda(r, r_1)\}}{(i\lambda r)^{\frac{1}{2}}\,(i\lambda r_1)^{\frac{1}{2}}} P_n(\cos \mathrm{POQ}), \quad . \quad (14)$$

where

$$p_{n+\frac{1}{2}}\{i\lambda(r, r_1)\} = K_{n+\frac{1}{2}}(i\lambda r)\, I_{n+\frac{1}{2}}(i\lambda r_1), \quad r > r_1. \quad (15)$$

When $r < r_1$, $p_{n+\frac{1}{2}}\{i\lambda(r, r_1)\}$ is to be replaced by $p_{n+\frac{1}{2}}\{i\lambda(r_1, r)\}$ on the right-hand side of (14). The integrations with respect to ϕ required in (11) as modified by (14) are known, being given by

$$\int_0^{\pi} d\phi\, P_n(\cos \mathrm{POQ}) = \pi P_n(\cos\theta) P_n(\cos\psi). \quad . \quad (16)$$

With the aid of (14) and (16) we accordingly obtain the solution representing a periodic circular line source of strength qe^{ikt} per unit length of the circle having centre O′ and radius $r_1 \sin\psi$ in the form

$$v = \frac{qr_1 \sin\psi e^{ikt}}{\mathrm{K}(rr_1)^{\frac{1}{2}}} \sum_0^\infty (n+\tfrac{1}{2}) p_{n+\frac{1}{2}}\{i\lambda(r, r_1)\}\, \mathrm{P}_n(\cos\theta)\, \mathrm{P}_n(\cos\psi) \quad . \quad . \quad . \quad (17)$$

for the case $r > r_1$. When $r < r_1$ the same expression with r and r_1 interchanged in $p_{n+\frac{1}{2}}$ represents the solution.

Periodic Spherical Surface Sources.

From the solution representing a circular line source of strength qe^{ikt} per unit length of the line, in the form (17), we may readily pass to the solutions representing a spherical surface source of strength $q\mathrm{P}_n(\cos\psi)e^{ikt}$ per unit area of the spherical surface of radius r_1. We have merely to replace q in (17) by $q\mathrm{P}_n(\cos\psi)r_1 d\psi$, and integrate the modified expression with respect to ψ from 0 to π. When we employ the notation $\mu = \cos\psi$ we find that the integrations required in carrying out the above process are

$$\int_{-1}^{+1} d\mu \mathrm{P}_n(\mu)\mathrm{P}_m(\mu) = 0\,; \quad \text{and} \quad \int_{-1}^{+1} d\mu \{\mathrm{P}_n(\mu)\}^2 = \frac{2}{2n+1} \quad . \quad . \quad . \quad (18)$$

and that the spherical surface source solution required is

$$v = \frac{qr_1^2 e^{ikt}}{\mathrm{K}(rr_1)^{\frac{1}{2}}} \mathrm{K}_{n+\frac{1}{2}}(i\lambda r)\, \mathrm{I}_{n+\frac{1}{2}}(i\lambda r_1)\mathrm{P}_n(\cos\theta) \quad . \quad . \quad (19)$$

for $r > r_1$, with the same expression with r and r_1 interchanged in $p_{n+\frac{1}{2}}$ for $r < r_1$. In this the strength per unit area of the surface $r = r_1$ is $qe^{ikt}\mathrm{P}_n(\cos\psi)$: we fall back upon the uniform spherical surface source of strength qe^{ikt} by putting $n = 0$ in (19). These results have been obtained earlier by a direct method, thus confirming (17).

Periodic Cylindrical Surface Sources.

In order to pass from (17) to solutions referring to a cylinder it is convenient to take the origin at the centre of the circular line regarded as source of strength qe^{ikt} per unit length. With ρ_1 to represent the radius of the

circle and ψ taken as $\frac{\pi}{2}$ the origin then coincides with the centre of the circle, and the solution (17) becomes, for the case $r > \rho_1$,

$$v = \frac{q\rho_1 e^{ikt}}{K(r\rho_1)^{\frac{1}{2}}} \sum_0^\infty (n+\tfrac{1}{2})\, p_{n+\frac{1}{2}}\{i\lambda(r,\, \rho_1)\}\, P_n(\cos\theta).P_n(0). \quad . \quad (20)$$

To pass from this to a cylindrical surface source of strength qe^{ikt} per unit area of the cylinder of radius ρ_1 replace q in (20) by $q\,dz$, r by $\sqrt{\rho^2+z^2}$, and integrate the above with respect to z over the length of the cylinder. For an infinite cylinder the integration extends from $-\infty$ to $+\infty$. When the point P $(\rho, 0)$, at which the effect of the cylindrical surface source is required, is external to the cylinder $(\rho > \rho_1)$, the integrals required form a series of which a typical term is represented by

$$I_n = \int_{-\infty}^{+\infty} dz\, T_n, \quad \text{where} \quad T_n = \frac{K_{n+\frac{1}{2}}(i\lambda r)}{r^{\frac{1}{2}}} P_n(\cos\theta). \quad (21)$$

The successive terms T_0, T_1, etc. obey the serial relation *

$$\frac{d}{dz} T_n = -\frac{i\lambda}{2n+1}[(n+1)T_{n+1} + nT_{n-1}], \quad . \quad (22)$$

which enables us to express each successive integral of the series in terms of the first integral alone denoted by I_0 where †

$$I_0 = 2\int_0^\infty dz \frac{K_{\frac{1}{2}}\{i\lambda\sqrt{\rho^2+z^2}\}}{(\rho^2+z^2)^{\frac{1}{4}}} = \left(\frac{2\pi}{i\lambda}\right)^{\frac{1}{2}} K_0(i\lambda\rho). \quad . \quad (23)$$

In virtue of the serial relation (22) we accordingly find

$$I_n = 2\int_0^\infty dz \frac{K_{n+\frac{1}{2}}\{i\lambda\sqrt{\rho^2+z^2}\}}{(\rho^2+z^2)^{\frac{1}{4}}} P_n(\cos\theta) = I_0 P_n(0). \quad (24)$$

Odd values of n in (20) disappear, since when n is odd $P_n(0)=0$. The solution representing the effect of an infinite cylindrical source for the case of a point $P(\rho, 0)$ external to the cylinder may now be put into the form

$$v = \frac{q\rho_1 e^{ikt}}{K} \cdot \left(\frac{2\pi}{i\lambda}\right)^{\frac{1}{2}} K_0(i\lambda\rho) \sum_0^\infty (n+\tfrac{1}{2}) \frac{I_{n+\frac{1}{2}}(i\lambda\rho_1)}{\rho_1^{\frac{1}{2}}} [P_n(0)]^2. \quad . \quad . \quad . \quad (25)$$

* This relation is stated inaccurately in the paper referred to above (on p. 922 of this paper).

† Watson's 'Bessel Functions,' p. 417.

There is no difficulty in verifying that this expression is equivalent to

$$v=\frac{q\rho_1 e^{ikt}}{K}\,K_0(i\lambda\rho)\,I_0(i\lambda\rho_1)\ldots(\rho>\rho_1),\quad .\quad .\quad (26)$$

the form given in an earlier investigation *.

When the effect of the cylindrical surface source is required at a point P within the cylinder ($\rho<\rho_1$), in the integration of (20) with respect to z, the two cases $r<\rho_1$ and $r>\rho_1$ must be taken into consideration, and the integrals required are

$$\int_0^{\cdot} dz\,\frac{I_{n+\frac{1}{2}}(i\lambda r)}{r^{\frac{1}{2}}}\,P_n(\cos\theta)\quad\text{and}\quad\int_r^{\infty} dz\,\frac{K_{n+\frac{1}{2}}(i\lambda r)}{r^{\frac{1}{2}}}\,P_n(\cos\theta).$$

$$.\quad .\quad .\quad (27)$$

These evaluations can be obtained, but in this case and in the corresponding cases relating to plane surface sources certain difficulties arise which are meantime reserved for further investigation.

Periodic Plane Surface Sources.

The point P at which the effect of a plane surface source is required may be taken on the axis of z drawn perpendicular to the plane, the origin being a point in the plane. The effect at P $(z, 0)$ of a circular line source of radius ρ_1 is given by

$$v=\frac{q\rho_1 e^{ikt}}{2K}\cdot\sqrt{\frac{2}{\pi}}\cdot i\lambda\cdot\frac{K_{\frac{1}{2}}(i\lambda r')}{(i\lambda r')^{\frac{1}{2}}},\quad\text{where}\quad r'^2=\rho_1^2+z^2.\quad (28)$$

Now replace q by $qd\rho_1$ and integrate the above expression with respect to ρ_1 between limits specified according as the source effect required is that due to a flat ring or a disk or an infinite plane surface. For a disk of radius a, acting as a periodic source of strength qe^{ikt} per unit area, the effect at any point situated on the axis of the disk is expressed by

$$v=\frac{qe^{ikt}}{2Ki\lambda}\{e^{-i\lambda z}-e^{-i\lambda R}\},\quad\text{where}\quad R^2=a^2+z^2,\quad .\quad (29)$$

which yields the infinite plane surface solution when a and R are both infinitely great.

* See also (41) below.

When the effect of a flat ring or disk source is required at any point P (r, θ), not on the axis of symmetry, the required solution may be arrived at by the following method. Starting with the circular line source solution given above in (28) for a point on the axis of the circle, first expand this expression by means of the addition theorem for Bessel functions, taking into consideration the two cases $z > \rho_1$ and $z < \rho_1$. This yields for $z > \rho_1$

$$v = \frac{q\rho_1 e^{ikt}}{K(z\rho_1)^{\frac{1}{2}}} \sum_0^\infty (n+\tfrac{1}{2})\, p_{n+\frac{1}{2}}\{i\lambda(z, \rho_1)\}\, P_n(0). \quad . \quad (30)$$

The effect at any point P (r, θ) may now be obtained from this by replacing z by r and multiplying each term by $P_n(\cos\theta)$, in accordance with a well-known theorem. In this way we recover the circular line solution (20), and the flat ring or disk source solutions may be obtained from (20) or from (30) by the usual integrations with respect to ρ_1.

Periodic Disk Source.

Consider the case of a disk of radius a acting as a heat source of strength qe^{ikt} per unit area. The solution is found by integrating equation (20) above. If the point P (r, θ) is situated so that $r > a$ the integrations required are indicated by the typical term

$$\int_0^a d\rho_1\, \rho_1^{n+1} t_n(\rho_1) = \left\{ \frac{a_n}{(i\lambda)} t_{n-1}(a) - \frac{na^{n-2}}{(i\lambda)^2} t_{n-2}(a) + \frac{n(n-2)a^{n-4}}{(i\lambda)^3} t_{n-3}(a) - \text{etc.} \right\}$$

$$-(-1)^{n/2} \frac{n(n-2)(n-4)\ldots 2}{(i\lambda)^{\frac{n+2}{2}}} t_{n-2}(\rho)_{\rho=0}, \quad (31)$$

where $$t_n(\rho) = \frac{I_{n+\frac{1}{2}}(i\lambda\rho)}{\rho^{n+\frac{1}{2}}}.$$

If the point P is so situated that $r < a$, the effect of the disk source of radius a at P can be obtained by taking the known effect at P due to a complete infinite plane source and deducting therefrom the effect of an infinite annulus source having the edge of the disk $(\rho = a)$ as its

inner boundary. The integrations required in carrying out this process are indicated by the typical term

$$\mathrm{I}_n{}' = \int_a^\infty d\rho_1 {\rho_1}^{n+1} t_n(\rho_1) = \left\{ \frac{a^n}{i\lambda} t_{n-1}(a) + \frac{n a^{n-2}}{(i\lambda)^2} t_{n-2}(a) + \frac{n(n-2)}{(i\lambda)^3} t_{n-3}(a) + \text{etc.} \right\}, \quad (32)$$

where $$t_n(\rho) = \frac{\mathrm{K}_{n+\frac{1}{2}}(i\lambda\rho)}{\rho^{n+\frac{1}{2}}}.$$

The solution representing the effect at P of the disk source of radius a and of strength qe^{ikt} per unit area is then given for $(r<a)$ by

$$v = \frac{qe^{ikt}}{2\mathrm{K}i\lambda} e^{-i\lambda z} - \frac{qe^{ikt}}{\mathrm{K}} \sum_0^\infty (n+\tfrac{1}{2}) \mathrm{I}_n{}' \frac{\mathrm{I}_{n+\frac{1}{2}}(i\lambda r)}{r^{\frac{1}{2}}} \mathrm{P}_n(\cos\theta)\, \mathrm{P}_n(0). \quad . \quad . \quad . \quad (33)$$

The alternative direct method of dealing with this case involves an integration of the type (31) from 0 to r, followed by an integration of the type (32) from r to a, and presents mathematical difficulties already referred to in the corresponding case of the cylindrical surface source.

Instantaneous Sources.

Corresponding to each solution representing a periodic source we can obtain another solution representing an instantaneous source, and still another solution representing a continuous source. For example, to the solution v_p, representing a periodic circular line source of strength qe^{ikt} per unit length of a given circle, there corresponds a solution v_i, representing an instantaneous circular line source of strength q per unit length of the circle, and another solution v_c, representing a constant circular line source of strength q per unit length of the circle per unit time. The process by which the instantaneous solution can be derived from the corresponding periodic solution in any particular case is indicated by

$$v_i = \frac{1}{\pi} \int_0^\infty dk\, v_p \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Instantaneous Circular Line Sources.

In this case we take v_p as the solution given in equation (20) above, and the integrations with respect to k can be performed for each term of the summation. We have

$$\int_0^\infty dk\, e^{\iota k t}\, p_{n+\frac{1}{2}}\{i\lambda(r,\rho_1)\} = \frac{\pi}{2t}\, e^{-\frac{r^2+\rho_1{}^2}{4\kappa t}}\, \mathrm{I}_{n+\frac{1}{2}}(\alpha) : \alpha = \frac{r\rho_1}{2\kappa t}, \quad . \quad . \quad . \quad (35)$$

in accordance with a result obtained in a former paper *. Hence we find that one form of the solution representing an instantaneous heat supply of amount q per unit length of the circle of radius ρ_1 is given by

$$v = \frac{q\rho_1\, e^{-\frac{r^2+\rho_1{}^2}{4\kappa t}}}{2\mathrm{K}t(r\rho_1)^{\frac{1}{2}}} \sum_0^\infty (n+\tfrac{1}{2})\mathrm{I}_{n+\frac{1}{2}}(\alpha)\, \mathrm{P}_n(\cos\theta)\, \mathrm{P}_n(0). \quad (36)$$

This result is derived ultimately from the periodic point source solution given in (6) above by two integrations—one with respect to ϕ along the circle of radius ρ_1 and another with respect to k. The result is obtained in a slightly different form by reversing the order of the two integrations. From (6) we obtain the instantaneous point source solution by the process given in (34) above :

$$v = \frac{q\kappa}{8\mathrm{K}(\pi\kappa t)^{3/2}}\, e^{-\frac{r^2}{4\kappa t}}. \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Taking the notation of p. 925, with the origin at the centre of the circle, we can express the instantaneous circular line source solution in the form

$$v = \frac{q\kappa\rho_1}{4\mathrm{K}(\pi\kappa t)^{3/2}} \int_0^\pi d\phi\, e^{-\frac{r'^2}{4\kappa t}} : r'^2 = r^2 + \rho_1{}^2 - 2r\rho_1 \cos(\mathrm{POQ}). \quad . \quad . \quad . \quad (38)$$

Here OP=O′P=r, PQ=r', and $\cos(\mathrm{POQ}) = \sin\theta\cos\phi$, since we have taken the origin to coincide with O′. The solution accordingly is

$$v = \frac{q\kappa\rho_1\, e^{-\frac{r^2+\rho^2}{4\kappa t}}}{4\mathrm{K}(\pi\kappa t)^{3/2}} \int_0^\pi d\phi\, e^{\beta\cos\phi}, \quad \text{where} \quad \beta = \left(\frac{\rho\rho_1}{2\kappa t}\right), \quad (39)$$

and we also have $\pi\mathrm{I}_0(\beta)$ as the value of the integral with respect to ϕ.

* Phil. Mag. xii. p. 238 (Suppl. Aug. 1931).

We can recover the equivalent form of this solution given in (36) above by using the expansion

$$e^{z\cos\psi}=(2\pi)^{\frac{1}{2}}\sum_{0}^{\infty}(n+\tfrac{1}{2})\frac{\mathrm{I}_{n+\frac{1}{2}}(z)}{z^{\frac{1}{2}}}\mathrm{P}_n(\cos\psi), \quad . \quad (40)$$

where $z=\alpha$ and $\psi=<\mathrm{POQ}$, in (38) before carrying out the integration with respect to ϕ. Since $\beta=\alpha\sin\theta$, the two equivalent solutions (36) and (39) yield the relation

$$\mathrm{I}_0(\alpha\sin\theta)=(2\pi)^{\frac{1}{2}}\sum_{0}^{\infty}(n+\tfrac{1}{2})\frac{\mathrm{I}_{n+\frac{1}{2}}(\alpha)}{(\alpha)^{\frac{1}{2}}}\mathrm{P}_n(\cos\theta)\,\mathrm{P}_n(0), \quad (41)$$

which is a more general relation than that provided by (25) and (26) above.

The case of the instantaneous circular line source for which the instantaneous supply per unit length of the line is $q\cos n(\phi+\phi_0)$ may be treated by either of the above methods. The angle ϕ_0 defines the position of the point A shown on the diagram on p. 925, A being the point at which the supply per unit length has its maximum value. Instead of (38) we now have

$$v=\frac{q\kappa\rho_1}{4\mathrm{K}(\pi\kappa t)^{3/2}}\int_0^{\pi}d\phi\cos n(\phi+\phi_0)e^{-\frac{r'^2}{4\kappa t}}, \quad . \quad (42)$$

or

$$v=\frac{q\kappa\rho_1\pi\cos n\phi_0}{4\mathrm{K}(\pi\kappa t)^{3/2}}e^{-\frac{r^2+\rho_1^2}{4\kappa t}}\mathrm{I}_n(\beta). \quad . \quad . \quad . \quad (43)$$

Instantaneous Cylindrical Surface Sources.

Consider an infinite cylindrical surface source for which the instantaneous supply is q per unit area at the surface $\rho=\rho_1$. To obtain this solution we may employ either form of the instantaneous line source solution (36) or (39). Replace q by $q\,dz$ and r by $(\rho^2+z^2)^{\frac{1}{2}}$, and integrate with respect to z. We may also use the method of (34), starting from the corresponding periodic solution given in (26). The latter method gives

$$v=\frac{q\rho_1}{2\mathrm{K}t}e^{-\frac{\rho^2+\rho_1^2}{4\kappa t}}\mathrm{I}_0(\beta) \quad . \quad . \quad . \quad . \quad . \quad (44)$$

as the solution representing the effect of an instantaneous supply q per unit area of the surface $\rho=\rho_1$.

When the instantaneous supply is $q\cos n(\phi+\phi_0)$ in-

stead of q the former method, by integrating (43) with respect to z, yields

$$v=\frac{q\rho_1\cos n\phi_0}{2\mathrm{K}t}e^{-\frac{\rho^2+\rho_1^2}{4\kappa t}}\mathrm{I}_n(\beta). \quad . \quad . \quad . \quad (45)$$

Instantaneous Plane Surface Sources.

The solution representing the effect of a flat disk of radius a as an instantaneous source of strength q per unit area may be obtained by any one of the various methods indicated above, but perhaps most readily by means of the instantaneous circular line source solution. In (39) replace q by $qd\rho_1$, and integrate with respect to ρ_1 from $\rho_1=0$ to $\rho_1=a$. This gives the required result in the form

$$v=\frac{q\kappa\pi}{4\mathrm{K}(\pi\kappa t)^{3/2}}\int_0^a d\rho_1\,\rho_1 e^{-\frac{r^2+\rho_1^2}{4\kappa t}}\mathrm{I}_0(\beta). \quad . \quad . \quad (46)$$

When a is made infinitely great we recover the solution representing an infinite plane source of strength q per unit area by an evaluation given in Watson's 'Bessel Functions,' p. 394.

The Continuous Circular Line Source.

As in the cases previously discussed the continuous circular line source solution provides the means of obtaining solutions for many continuous surface sources. It can be directly formed with the aid of the contiuuous point source solution

$$v=\frac{q}{4\pi\mathrm{K}r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Considering the circular line of radius ρ_1 as a continuous source supplying amount q per unit length per unit time, we find

$$v=\frac{q\rho_1}{2\mathrm{K}(\rho_1^2+z^2)^{\frac{1}{2}}}$$

as the effect at any point $(z, 0)$ on the axis of the circle. The effect at any point P (r, θ) is then obtained by means of the well-known theorem already employed above. For $r<\rho_1$ and $r>\rho_1$ the solutions are respectively

$$v=\frac{q}{2\mathrm{K}}\left\{1+\frac{r^2}{\rho_1^2}\mathrm{P}_2(\cos\theta)\,\mathrm{P}_2(0)\right.$$
$$\left.+\frac{r}{\rho_1^4}\mathrm{P}_4(\cos\theta)\,\mathrm{P}_4(0)+\text{etc.}\right\}, \quad . \quad (48)$$

$$v = \frac{q\rho_1}{2\mathrm{K}r}\left\{1+\frac{\rho_1^2}{r^2}\mathrm{P}_2(\cos\theta)\,\mathrm{P}_2(0) + \frac{\rho_1^4}{r^4}\mathrm{P}_4(\cos\theta)\,\mathrm{P}_4(0)+\text{etc.}\right\}. \quad . \quad (48')$$

It is of interest to derive this from instantaneous solution (35), or (39). In (36) replace q by qdt' and t by $(t-t')$, and integrate with respect to t' from $t'=0$ to $t'=t$, and then let t become infinitely great. The integrations required in the application of this process are represented by

$$\int_0^\infty dz\, \frac{e^{-az}\mathrm{J}_{n+\frac{1}{2}}(bz)}{z} = \frac{\{(a^2+b^2)^{\frac{1}{2}}-a\}^{n+\frac{1}{2}}}{(n+\frac{1}{2})\,b^{n+\frac{1}{2}}}, \quad . \quad (49)$$

where $$a=\frac{r^2+\rho_1^2}{4\kappa}, \quad ib=\frac{r\rho_1}{2\kappa}.$$

These yield a result in agreement with the solution (48). In like manner each continuous source solution of the equation $\nabla^2(v)=0$ can be derived from a corresponding solution of equation (2) above, and many applications and evaluations of integrals involving Bessel Functions can thereby be obtained.

LXXV. *On a Fundamental Problem in Diffraction.* By George Green, *D.Sc., Lecturer in Physics in the Applied Physics Department of the University of Glasgow* *.

THE problem chosen for consideration is the first of a group of diffraction problems for which approximate solutions have been given by the late Lord Rayleigh in 1897 † and in later papers. It may be stated shortly as follows :—

Plane waves are incident normally upon a plane screen pierced by a small circular aperture, and it is required to determine the resultant wave motion on either side of the screen.

The problem, which is treated by the late Lord Rayleigh in its special relation to sound, has important optical and

* Communicated by the Author.

† 'Scientific Papers,' iv. p. 283, and vi. p. 161 ; 'Theory of Sound,' vol. ii. § 292 and § 306.

other analogues. In the discussion given below it is for convenience treated as a problem in heat conduction. In the fluid motion analogue the condition of a vanishing normal velocity over the screen must be fulfilled, and the essence of the problem is to determine the distribution of fluid velocity across the aperture. In the corresponding heat conduction problem the condition becomes that of a vanishing normal gradient of temperature over the screen, and the essence of the problem is to determine the distribution of temperature gradient across the aperture. Lord Rayleigh's approximate solution of the problem applies to the case in which the waves incident upon the screen are of exceedingly low frequency. The fluid motion through the aperture is practically a steady motion, the frequency term being negligible *. The purpose of the present paper is to indicate a method whereby the distribution of the gradient of temperature across the aperture can be determined at any frequency, the condition of a zero gradient upon the screen being secured by successive temperature distributions applied to the screen.

Starting from the fundamental equation of heat conduction,

$$\frac{dv}{dt}=\kappa\nabla^2(v), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and applying the condition that v varies with time according to the factor e^{ikt}, we obtain the differential equation to be solved in the form

$$(\nabla^2+\lambda^2)v=0, \quad \text{where } i\lambda=\sqrt{\frac{ik}{\kappa}}, \quad . \quad . \quad (2)$$

in the notation employed in previous papers. As source or origin of the plane waves we may take the plane $z=0$ considered as a heat source of strength qe^{ikt} per unit area. That is, we take the solution

$$v=\frac{q}{2\mathrm{K}i\lambda}e^{ikt-i\lambda z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

to represent the wave incident upon a screen impervious to heat facing the source at the plane $z=z_1$. Provided that the screen be complete the condition to be fulfilled

* See also Richardson & Tyler, Phys. Soc. Proc. p. 5 (Dec. 1929).

at $z=z_1$ is $\frac{dv}{dz}=0$, and the required solution, which now includes a reflected wave, is

$$v=\frac{qe^{ikt}}{2\mathrm{K}i\lambda}\{e^{-\lambda z}+e^{-i\lambda(2z_1-z)}\}. \quad . \quad . \quad . \quad . \quad (4)$$

This represents temperature conditions at all points in the region $z=0$ to $z=z_1$, there being no heat transmission across the screen.

On the screen the temperature is given by

$$v=2\theta_0 e^{ikt}=\frac{q}{\mathrm{K}i\lambda}e^{ikt-i\lambda z_1}, \quad . \quad . \quad . \quad . \quad (5)$$

where $\theta_0 e^{ikt}$ is the temperature at the screen due to the incident wave and also the temperature at the screen due to the reflected wave.

With reference to the solution (4) the origin plane may be referred to as a periodic heat source and the complete screen as a periodic temperature source. It is now required to express the effect represented by (4) in terms of effects originating at each element of area of each of the two boundary planes $z=0$ and $z=z_1$. This can be done by means of the solution representing a periodic point source of strength qe^{ikt} at the origin, given by

$$v=\frac{qe^{ikt}}{4\pi\mathrm{K}}\frac{e^{-i\lambda r}}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We may treat each element of area $d\mathrm{S}$ of the origin plane as a point source of strength $qd\mathrm{S}e^{ikt}$, and the solution representing the effect at any point P due to the infinite plane acting as a periodic heat source of strength qe^{ikt} per unit area is then obtained in the form

$$v=\frac{qe^{ikt}}{4\pi\mathrm{K}}\int^{\mathrm{S}} d\mathrm{S}\frac{e^{-i\lambda r'}}{r'}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in which r' represents the distance from any element $d\mathrm{S}$ to the point P. It may be sufficient at present to verify that this yields the correct result for a point P situated at z on the axis OZ. Taking $d\mathrm{S}=2\pi\rho\, d\rho$ and $r'^2=\rho^2+z^2$, we obtain from (7) the effect at P due to the complete plane $z=0$ in the form

$$v=\frac{qe^{ikt}}{2\mathrm{K}}\int_0^\infty d\rho\,\rho\frac{e^{-i\lambda r'}}{r'}=\frac{qe^{ikt}}{2\mathrm{K}i\lambda}e^{-i\lambda z}, \quad . \quad . \quad . \quad (8)$$

which is in agreement with the effect as given by solution (3).

The solution (7) should still be applicable when one element of area dS alone is acting as a periodic heat source. For dS as a heat source of strength qe^{ikt} per unit area we have the solution

$$v=\frac{qe^{ikt}}{4\pi K}\frac{e^{-i\lambda r}}{r}dS, \quad \ldots\ldots \quad (9)$$

which is one of two fundamental solutions associated with the element of area dS. It has the following properties :—

(*a*) It fulfils the condition of heat supply at dS (qe^{ikt} per unit area).

(*b*) It represents zero heat supply at all points in the plane of dS external to dS.

(*c*) It imposes a certain temperature at all points in that plane.

The second fundamental solution referred to may also be derived from (6). To ascertain its relation to (9) consider the solution

$$v=\theta_0 e^{ikt-i\lambda z}; \quad z>0 \quad \ldots\ldots \quad (10)$$

in its relation to the periodic plane heat source solution given in (3) above. The expression (10) satisfies the condition $v=\theta_0 e^{ikt}$ at the origin plane on the side $z>0$, and it is therefore the periodic plane temperature source solution. The solution (10) is obviously obtainable from the solution (3) by differentiation with respect to z. Thus the solution (10) is equivalent to

$$v=-2K\frac{d}{dz}v(3), \quad \ldots\ldots \quad (11)$$

q being replaced by θ_0 ; and this accordingly represents the plane periodic temperature source of strength $\theta_0 e^{ikt}$ on the positive side of the origin plane. Now apply the process indicated in (11) to the solution (9), and we obtain the solution representing the periodic temperature source of strength $\theta_0 e^{ikt}$ on the z positive side of the element of area dS in the form

$$v=-\frac{\theta_0 e^{ikt}}{2\pi}dS\frac{d}{dz}\left(\frac{e^{-i\lambda r'}}{r'}\right), \quad \ldots\ldots \quad (12)$$

r' being the distance from dS to any point P at which the temperature due to the source at dS is required. As before, we may readily verify for the special case where point P is on the axis OZ that by integration of (12) with respect to S over the whole origin plane we recover the solution (10) as required. Solution (12) is the second fundamental solution associated with the element of area dS at our disposal. Its properties are these :—

(*a*) It imposes the temperature $\theta_0 e^{ikt}$ at element dS of the plane.

(*b*) It imposes zero temperature at all other elements in the same plane.

(*c*) It imposes a temperature gradient at all points in the plane in the direction of the normal to the plane.

The solution (4) expresses the conditions that over the complete plane $z=0$ there is a heat supply of amount qe^{ikt} per unit area, and over the complete plane $z=z_1$, in addition to the temperature produced by the incident wave, the temperature $\theta_0 e^{ikt}$ is maintained. By means of the two fundamental solutions (9) and (12) the effect due to the incident wave originating at $z=0$ and the reflected wave originating at $z=z_1$ may now be expressed in terms of the conditions existing on the boundary planes for any point lying between the planes. If we now assume that there exists in the screen at $z=z_1$ a circular aperture, of radius a, having its centre on the axis OZ at point O′ where the axis meets the plane $z=z_1$, the effect at any point between the boundary planes, and therefore at any point close to the aperture itself, is still determined by the conditions existing on the boundary planes. We have seen that for a complete screen the reflected wave is that due to a temperature $\theta_0 e^{ikt}$ maintained over the complete plane. We may therefore consider first the effect produced at any point within the aperture due to $\theta_0 e^{ikt}$ maintained over the actual screen.

Effect at any Point due to Temperature $\theta_0 e^{ikt}$ maintained over the Screen.

To express this effect in analytical form it is convenient to take the origin at O′, the centre of the aperture and

the axis O′Z′ towards the origin O—that is, in the direction opposite to OZ. Here (z', ρ) are coordinates of any point referred to O′, and the circular line source solution representing a periodic heat supply qe^{ikt} per unit length of the circle of radius ρ' is given by

$$v=\frac{qe^{ikt}\rho'}{\mathrm{K}(r\rho')}\sum_0^\infty (n+\tfrac{1}{2})p_{n+\frac{1}{2}}\{i\lambda(\rho', r)\}\,\mathrm{P}_n(\cos\theta)\,\mathrm{P}_n(0) \quad (13)$$

(see equation (20) of preceding paper). We are here considering the case where $\rho'>r$—for any point within the aperture $(r<a)$—so that

$$p_{n+\frac{1}{2}}\{i\lambda(\rho', r)\}=\mathrm{K}_{n+\frac{1}{2}}(i\lambda\rho')\mathrm{I}_{n+\frac{1}{2}}(i\lambda r). \quad . \quad . \quad (14)$$

The effect at r due to the heat distribution qe^{ikt} per unit area on the whole screen extending from $\rho=a$ to $\rho=\infty$ is represented by

$$v=\frac{qe^{ikt}}{\mathrm{K}}\sum_0^\infty (n+\tfrac{1}{2})\mathrm{I}'_n\mathrm{T}_n\mathrm{P}_n(0), \quad . \quad . \quad . \quad (15)$$

where

$$\mathrm{I}'_n=\int_a^\infty d\rho'\,\rho'\frac{\mathrm{K}_{n+\frac{1}{2}}(i\lambda\rho')}{\rho'^{\frac{1}{2}}}, \quad \text{and} \quad \mathrm{T}_n=\frac{\mathrm{I}_{n+\frac{1}{2}}(i\lambda r)}{r^{\frac{1}{2}}}\mathrm{P}_n(\cos\theta). \quad . \quad . \quad . \quad (15')$$

The corresponding solution representing the effect at r due to a temperature distribution $\theta_0 e^{ikt}$ over the whole screen $(\rho>a)$ is, in accordance with (11) above, given by

$$v=-2\theta_0 e^{ikt}\sum_0^\infty (n+\tfrac{1}{2})\mathrm{I}'_n\frac{d\mathrm{T}_n}{dz'}\mathrm{P}_n(0). \quad . \quad . \quad (16)$$

The successive terms T_n obey the serial relation

$$(2n+1)\frac{d\mathrm{T}_n}{dz'}=i\lambda\{(n+1)\mathrm{T}_{n+1}+n\mathrm{T}_{n-1}\}. \quad . \quad (17)$$

The gradient of temperature at any point on the aperture $(z'=0,\ \rho<a,\ \cos\theta=0)$ is now readily obtainable from (16) and (17). If we introduce the notation

$$\mathrm{E}_n=\left\{\frac{d^2}{dz'^2}\mathrm{T}_n\mathrm{P}_n(0)\right\}_{z'=0}(n+\tfrac{1}{2}) \quad . \quad . \quad . \quad (18)$$

we find that, in virtue of the relation $(n+2)P_{n+2}(0) = -(n+1)P_n(0)$ and of a well-known theorem relating to the Bessel Functions involved, we may also write

$$E_n = \frac{i\lambda}{\rho^{3/2}} \{(n+1)^2 I_{n+\frac{1}{2}}(i\lambda\rho) - n^2 I_{n-\frac{1}{2}}(i\lambda\rho)\} [P_n(0)]^2, \quad (19)$$

and the gradient at any point, ρ, in the aperture is then expressed by

$$-\frac{dv}{dz'}_{(z'=0)} = \theta_0 e^{ikt} \sum_0^\infty I_n' E_n. \quad . \quad . \quad . \quad (20)$$

Evaluations of the integrals (I_n') are given in (32) of the preceding paper, by means of which we obtain the following results :—

$$\left.\begin{aligned} I_0' &= \sqrt{\frac{\pi}{2}}\, e^{-i\lambda a}/(i\lambda)^{3/2}, \\ I_2' &= I_0' \left\{1 + \frac{3}{i\lambda a}\right\}, \\ I_4' &= I_0' \left\{1 + \frac{10}{(i\lambda a)} + \frac{35}{(i\lambda a)^2} + \frac{35}{(i\lambda a)^3}\right\}, \\ I_6' &= I_0' \left\{1 + \frac{16}{(i\lambda a)} + \frac{105}{(i\lambda a)^2} + \frac{315}{(i\lambda a)^3} + \frac{315}{(i\lambda a)^4}\right\}. \end{aligned}\right\} \quad (21)$$

These results enable us to arrange the expression representing the gradient in a series of ascending powers of λ : the first term, which is independent of λ as multiplier, is then given by

$$-\frac{dv}{dz'}_{(z'=0)} = \theta_0 e^{ikt} \frac{e^{-i\lambda a}}{a}\left(1 + \frac{3}{4}\left(\frac{\rho}{a}\right)^2 + \frac{45}{64}\left(\frac{\rho}{a}\right)^4 + \ldots\right) + \cdot \quad . \quad . \quad . \quad (22)$$

This expression represents the distribution of the gradient values across the aperture in the case where the period of the temperature variation on the screen is exceedingly long—that is, $\lambda \to 0$ and $k \to 0$. If we postulate a steady temperature θ_0 over the screen, and seek to determine the corresponding distribution of the gradient across the aperture directly, we have the solution, equivalent to (12) with $k=0$, and $\lambda=0$,

$$v = -\frac{\theta_0}{2\pi}\int^S dS \frac{d}{dz}\cdot\frac{1}{r}, \quad . \quad . \quad . \quad . \quad (23)$$

S being the area of the screen, which yields again the value for the gradient given by (22) when k and λ both vanish.

The above investigation illustrates the general theorem that the distribution of the gradient across a plane on which is imposed a periodic temperature distribution is in the limit the same as that produced by the same temperature distribution applied steadily to the plane. This theorem, which here refers to distributions which are symmetrical about a point in the plane, may readily be proved to hold in the case of a circular line source situated at any radius ρ' on the plane. It must therefore be valid in the general case where the temperature distribution on the screen is given by $f(\rho')e^{ikt}$, $(k \to 0)$. As the steady temperature distributions in each case are derivable from corresponding periodic distributions by a process indicated in the preceding paper the theorem is worthy of more complete examination than is possible at present.

Returning again to the solution (4) in its relation to the aperture problem we note that the incident wave represented by (3) above imposes at all points in the screen the temperature $\theta_0 e^{ikt}$, and the gradient in the positive direction of z represented by $-\frac{dv}{dz} = i\lambda\theta_0 e^{ikt}$. When the screen is without aperture the reflected wave is that due to imposing the temperature $\theta_0 e^{ikt}$ over the complete screen. This temperature distribution completely annuls the gradient due to the incident wave at all points in the plane of the screen. When the screen contains an aperture of radius a, and the distribution $v = \theta_0 e^{ikt}$ is maintained over the actual screen only, the reflected wave as represented by this condition can no longer completely annul the gradient produced at each point of the screen by the incident wave. There remains unannulled at each point a gradient equal to that which is produced at each point of the screen by imposing the condition $v = \theta_0 e^{ikt}$ over the area of the aperture of radius a in the plane of the screen. The distribution $\theta_0 e^{ikt}$ over the screen annuls the incident wave gradient in part. The problem is now to determine that temperature distribution $v = f(\rho)e^{ikt}$ which, imposed upon the screen, produces at each point of the screen a gradient

equal to that produced by the temperature distribution $v=\theta_0 e^{ikt}$ imposed over the aperture.

Effect of Temperature $v=\theta_0 e^{ikt}$ imposed over the Aperture.

The solution representing this effect is similar to (16), and is obtained in exactly the same manner. For any point such that $r>a$ it may be written in the form

$$v=-2\theta_0 e^{ikt}\sum_0^\infty (n+\tfrac{1}{2})\mathrm{I}_n'\frac{d\mathrm{T}_n}{dz'}\mathrm{P}_n(0), \quad . \quad . \quad (24)$$

where

$$\mathrm{I}_n'=\int_0^a d\rho'\,\rho'^{\frac{1}{2}}\mathrm{I}_{n+\frac{1}{2}}(i\lambda\rho') \quad \text{and} \quad \mathrm{T}_n=\frac{\mathrm{K}_{n+\frac{1}{2}}(i\lambda r)}{r^{\frac{1}{2}}}\mathrm{P}_n(\cos\theta). \quad . \quad . \quad . \quad (25)$$

From this we can find the value of the gradient at any point $\rho>a$ on the screen. In this case instead of (17) we have

$$(2n+1)\frac{d\mathrm{T}_n}{dz'}=-i\lambda\{(n+1)\mathrm{T}_{n+1}+n\mathrm{T}_{n-1}\}, \quad . \quad (26)$$

and for E_n, defined as in (18) above, we have now the value

$$\mathrm{E}_n=\frac{i\lambda}{\rho^{3/2}}\{-(n+1)^2\mathrm{K}_{n+\frac{3}{2}}(i\lambda\rho)+n^2\mathrm{K}_{n-\frac{1}{2}}(i\lambda\rho)\}[\mathrm{P}_n(0)]^2. \quad (27)$$

With this notation the expression representing the gradient at any point, ρ, on the screen, due to the temperature distribution $\theta_0 e^{ikt}$ over the aperture, is in form the same as (20) above. The integrals, I_n', are evaluated in (31) of the preceding paper, from which we obtain the values

$$\left.\begin{aligned}
\mathrm{I}_0'&=\mathrm{A}(i\lambda a)^2\left\{\frac{1}{2!}+\frac{(i\lambda a)^2}{4!}+\frac{(i\lambda a)^4}{6!}+\text{etc.}\right\},\\
\mathrm{I}_2'&=\mathrm{A}(i\lambda a)^4\left\{\frac{1}{3.4.5}+\frac{(i\lambda a)^2}{3.3.4.5.7}+\frac{(i\lambda a)^4}{64.1.3.5.7.9}\right\},\\
\mathrm{I}_4'&=\mathrm{A}(i\lambda a)^6\left\{\frac{1}{3.5.6.7.9}+\frac{(i\lambda a)^2}{16.1.3.5.7.9.11}\right\},\\
\mathrm{I}_6'&=\mathrm{A}(i\lambda a)^8\left\{\frac{1}{8.1.3.5.7.9.11.13}+\right\},
\end{aligned}\right\} \quad (28)$$

in which $$\mathrm{A}=\sqrt{\frac{2}{\pi}}\Big/(i\lambda)^{3/2}.$$

These results again enable us to arrange the expression representing the gradient perpendicular to the screen in a series of ascending powers of λ, the first terms being given by

$$-\frac{dv}{dz'}_{(z'=0)} = -\theta_0 e^{ikt-i\lambda\rho} \times \left\{\frac{a^2}{2\rho^3}\left(1+\frac{9}{8}\frac{a^2}{\rho^2}+\frac{75}{64}\frac{a^4}{\rho^4}+\text{etc.}\right)(1+i\lambda\rho)\right\}. \quad (29)$$

When in this expression we put $\lambda\to 0$ and $e^{ikt}\to 1$ we should again obtain the gradient perpendicular to the screen produced by imposing the uniform temperature θ_0 over the area of the aperture, in accordance with the theorem stated on p. 941. This we now proceed to confirm.

Effect of Temperature θ_0 imposed over the Aperture.

The solution representing this effect we can obtain directly by means of Fourier's integral theorem. If we put

$$v=\theta_0\int_0^\infty dk\, ke^{-kz'}\mathrm{J}_0(k\rho\int_0^a d\rho'\rho'\mathrm{J}_0(k\rho'), \quad . \quad (30)$$

the function v satisfies Laplace's equation at all points ($z'>0$), and on plane $z'=0$ it fulfils the conditions $v=\theta_0$ for $\rho<a$ and $v=0$ for $\rho>a$. Equation (30) therefore represents the required solution. It gives

$$v=\theta_0 a\int_0^\infty dk\, e^{-kz'}\mathrm{J}_0(k\rho)\mathrm{J}_1(ka) \quad . \quad . \quad . \quad (31)$$

and

$$-\frac{dv}{dz'}_{z'=0} = \theta_0 a\int_0^\infty dk\, k\mathrm{J}_0(k\rho)\mathrm{J}_1(ka), \quad \rho>a, \quad . \quad (32)$$

$$= -\theta_0\left\{\frac{a^2}{2\rho^3}\left(1+\frac{9}{8}\frac{a^2}{\rho^2}+\frac{75}{64}\frac{a^4}{\rho^4}+\text{etc.}\right)\right\}, \quad (32')$$

by an evaluation given in Watson's 'Bessel Functions,' p. 401.

As we have seen above the incident wave gradient perpendicular to the screen is partially annulled by the temperature distribution $\theta_0 e^{ikt}$ over the screen ($\rho>a$). It is now required to annul the remaining gradient, represented by (29) with sign changed, by imposing

an additional temperature distribution over the screen—that is to say, that the additional distribution over region ($\rho > a$) of the plane containing the screen, together with the distribution $\theta_0 e^{ikt}$ over the region ($\rho < a$) of the plane, must satisfy the condition $\frac{dv}{dz'} = 0$ at all points of the screen ($\rho > a$). For the direct determination of the expression representing the required distribution over the screen the two fundamental solutions (9) and (12) are available. These give the solution in an integral form; an equivalent solution can be found from (15), with q determined by (29), but this method of procedure involves the difficulty referred to on pp. 928 and 930 of the preceding paper. An alternative method, however, can be found, depending upon the theorem stated on p. 941, to give the most important term of the required result. For this it is first necessary to solve the problem of determining the steady temperature throughout the medium $z' > 0$, corresponding to the specified conditions on the plane, $z' = 0$,

$$\left.\begin{aligned} v &= \theta_0 \text{ over the region } \rho < a, \\ -\frac{dv}{dz'} &= 0 \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \rho > a. \end{aligned}\right\} \quad . \quad . \quad (33)$$

These conditions are identical with the conditions fulfilled by the potential in the case of an insulated electrified disk * occupying the position of the aperture and charged to potential $v = \theta_0$. The potential function in the problem of the electrified disk corresponds exactly with the temperature function in the problem of steady temperature distribution under consideration. According to the known solution of the disk problem the solution of Laplace's equation which satisfies the specified conditions gives in the plane of the aperture ($z' = 0$)

$$v = \theta_0 \frac{2}{\pi} \sin^{-1} \frac{a}{\rho} \text{ over the region } \rho > a. \quad . \quad (34)$$

This distribution of steady temperature on the screen, taken with $v = \theta_0$ over the aperture, would therefore produce the same temperature throughout the solid ($z' > 0$) as that determined by the two conditions stated in (33) above.

* Rayleigh, 'Theory of Sound,' ii. § 306.

Effect of Temperature $\theta_0 \cdot \frac{2}{\pi}\sin^{-1}\frac{a}{\rho}$ *imposed over the Screen.*

The Fourier integral solution in this case is

$$v=\frac{2\theta_0}{\pi}\int_0^\infty dk\, ke^{-kz'}\mathrm{J}_0(k\rho)\int_a^\infty d\rho'\,\rho'\sin^{-1}\frac{a}{\rho'}\mathrm{J}_0(k\rho'). \quad (35)$$

An integration by parts enables us to evaluate the first integral, and yields

$$v=-a\theta_0\int_0^\infty dk\, e^{-kz'}\mathrm{J}_0(k\rho)\mathrm{J}_1(ka) + \frac{2a\theta_0}{\pi}\int_0^\infty dk\, e^{-kz'}\mathrm{J}_0(k\rho)\frac{\sin ka}{k}. \quad (36)$$

Hence we obtain also (see Watson's 'Bessel Functions,' p. 405)

$$-\left(\frac{dv}{dz'}\right)_{z'=0}=-a\theta_0\int_0^\infty dk\, k\mathrm{J}_0(k\rho)\mathrm{J}_1(ka), \quad (\rho>a), \quad (37)$$

as required in order to annul the gradient (32) on the screen.

Effect of Temperature $\theta_0\frac{2}{\pi}\sin^{-1}\frac{a}{\rho}e^{ikt}$ *imposed over the Screen.*

The gradient at any point on the screen due to this distribution is approximately that required to annul at each point the gradient represented by (29) above. The annulment is confirmed above for the case $\lambda\to 0$, in accordance with the theorem stated on p. 941. If, for the case of λ finite, we assume that an additional distribution, $f(\rho\lambda)e^{ikt}$, is required to be imposed on the screen to secure the complete annulment and enquire into the conditions determining $f(\rho\lambda)$, it appears that these conditions include $f(\rho\lambda)\to 0$ for $\lambda\to 0$ and for for $\rho\to\infty$.

An expression representing $f(\rho\lambda)$ can readily be found in the form of a Fourier integral of the same type as the second integral in (36). This, however, will be studied later.

The Wave-system reflected from the Screen.

According to the above reasoning the wave-system reflected from the screen is approximately that produced by imposing on the screen on the side opposite to the original source the temperature distribution

$$v=\theta_0 e^{ikt}\left(1-\frac{2}{\pi}\sin^{-1}\frac{a}{\rho}\right)=\frac{2\theta_0}{\pi}e^{ikt}\cos^{-1}\frac{a}{\rho}. \quad . \quad (38)$$

The resultant temperature at any point of the screen is obtained by adding to this, $\theta_0 e^{ikt}$, the temperature due to the incident wave, thus making the temperature distribution continuous over the whole plane ($z'=0$), with values ranging from $\theta_0 e^{ikt}$ over the aperture to $2\theta_0 e^{ikt}$ at infinite distance from the aperture. On the other hand, the gradient of temperature perpendicular to the plane $z'=0$ is discontinuous over the plane. At the centre of the aperture the value is finite, and it increases to an infinite value at the edge of the aperture. On the screen its value is zero (or nearly zero) everywhere. The value of the gradient at each point of the aperture is the resultant of the gradient due to the incident wave, $i\lambda\theta_0 e^{ikt}$, and the gradient due to the reflected wave, (38) above.

Effect at any Point in the Aperture.

The incident wave-train produces at each point of the aperture the conditions

$$v=\theta_0{}^{ikt}: \quad -\frac{dv}{dz}=i\lambda\theta_0 e^{ikt},$$

where θ_0 is given by (3) at $z=z_1$.

The reflected wave-system produces zero temperature at all points of the aperture and a gradient determined by (38) above. The solution representing the reflected system is given in form by (16):

$$v=-2\theta_0 e^{ikt}\sum_0^\infty (n+\tfrac{1}{2})\mathrm{I}_n^l\frac{d\mathrm{T}_n}{dz'}\mathrm{P}_n(0), \quad . \quad . \quad (39)$$

where

$$\mathrm{I}_n^l=\frac{2\theta_0}{\pi}\int_a^\infty d\rho'\cos^{-1}\left(\frac{a}{\rho'}\right)\rho'^{\frac{1}{2}}\mathrm{K}_{n+\frac{1}{2}}(i\lambda\rho'), \quad . \quad (40)$$

and T_n has again the value given in (15′) above. The gradient at any point ρ in the aperture is again expressed

by (20), with I_n^l as stated in (40) replacing that stated in (15′), and with E_n still defined by (19).

The evaluation of the integrals contained in (40) must be deferred to a later opportunity for the more complete study of the distribution of the gradient across the aperture given in solution (39) for finite values of λ. Meantime it is of importance to confirm that, for the case $\lambda \to 0$, the equation (39) yields the value

$$-\frac{dv}{dz'}_{z'=0} = \theta_0 e^{ikt} \frac{2}{\pi a} \left\{ 1 + \frac{1}{2}\frac{\rho^2}{a^2} + \frac{3}{8}\frac{\rho^4}{a^4} + \right\} \quad . \quad (41)$$

$$= \theta_0 e^{ikt} \frac{2}{\pi\sqrt{a^2-\rho^2}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41')$$

Thus the first term of the gradient obtained from (39) is in agreement with the value given in Lord Rayleigh's approximate solution. It is accordingly possible by the above method to determine the distribution of the normal gradient across the aperture. The determination of the wave-system transmitted through the aperture is then possible by well-known methods.

LXXVI. *The Dual Nature of Modulation.* *By* E. Z. Stowell, *Associate Physicist, and* A. F. Deming, *Junior Electrical Engineer, Sound Laboratory, National Advisory Committee for Aeronautics, Langley Field, Virginia, U.S.A.* *

Abstract.

A brief *résumé* of the controversy regarding the reality of sidebands and the nature of amplitude modulation is given. The paper endeavours to establish the equivalence of the two points of view brought forth in the controversy, namely, that a modulated wave may consist either of a carrier of constant amplitude accompanied by sidebands or of a wave of a single frequency but with varying amplitude. The response of a resonant circuit is computed for both types of excitation and is found to be the same for each. Thus, a modulated wave may

* Communicated by the Authors.

be said to possess a dual nature ; either point of view may be adopted at will and either manifestation may be detected by suitable apparatus.

Practical examples of both points of view are given, and the conditions under which it is advantageous to adopt one point of view in preference to the other are discussed.

The Sideband Controversy.

A FEW years ago a discussion took place in the pages of ' Nature ' regarding the nature of modulation. The philosophical importance of the arguments warrants their exposition in some detail.

Sir Ambrose Fleming [1] opened the controversy with the statement of what we shall refer to hereafter as the " single-frequency " point of view. He declared the sidebands, which are supposed to come into existence as a result of modulation, to be mathematical fictions, artificial, and unnecessary to the discussion of a modulated wave. To Fleming a modulated wave simply meant a wave of the frequency of what radio engineers call the " carrier," but whose amplitude varies in accordance with the impressed modulation. Band width, he argued, has no meaning ; the restriction of a radio wave to a 9 kc. width is unnecessary, artificial, and scientifically unjustifiable. One should ask not how wide a band should be transmitted, but how much variation in amplitude is permissible. He likened the sideband situation to that in mathematical physics, in which a velocity is split into its Cartesian components not because the components have any real existence but simply for convenience in mathematical treatment.

C. L. Fortescue [2] replied, disagreeing with Fleming, and pointed out the experimental evidence for the real existence of sidebands. At the same time he agreed that a modulated wave could be thought of in the manner suggested by Dr. Fleming.

Mr. L. H. Bedford [2] put forth the interesting suggestion that a method of computing circuit response to a modulated wave might be found which would not employ the sideband concept, thus giving Fleming's point of view a mathematical as well as a philosophical existence.

Fleming [2], in reply, stated that there was no experimental evidence that radio signals possessed a definite band width, that the higher audio-frequencies were not perceptibly cut in amplitude, and that therefore restriction to a 9 kc. band width was unnecessary and artificial. He called attention to the desirability of using radio circuits of extreme selectivity.

Sir Oliver Lodge [3] then called Dr. Fleming's attention to the exponential or transient term in the solution of the circuit equation ; he thought that this term might be of importance with a wave of varying amplitude. He stated his belief in the physical existence of sidebands.

C. L. Fortescue [3] disputed the accuracy of Fleming's statements about selectivity, maintaining that the higher audio-frequencies were actually reduced in amplitude with selective receivers, and that the restriction to a definite band width was physically necessary.

J. A. Ratcliffe [3] called attention to the every-day fact of single sideband transmission in transatlantic telephony and to the work of Bown, Martin, and Potter, who transmitted a modulated wave, received the three components separately, and showed that the ionized layer treated the three components as though they were physically distinct.

R. T. Glazebrook [3] showed mathematically that response in a receiver tuned to a sideband would be expected if a modulated wave were impressed upon it.

Mr. G. B. Brown [3] felt that the experimental evidence for the existence of sidebands was overwhelming proof of their reality.

F. M. Colebrook [4] then brought forth experimental evidence, obtained with tuned circuits, that sidebands have a real existence in the neighbourhood of 40 kc. ; the peaks of circuit response occurred in the predicted positions.

Very recently Hazel [5] in this country has examined the whole question experimentally, and has once more demonstrated the physical reality of sidebands by detecting them in tuned circuits.

Thus, while the question of the physical reality of sidebands is settled, the controversy is still unfinished ; for, in spite of Dr. Fleming's error in regard to the cutting of sidebands, there still remains considerable justification for his point of view.

In the following discussion, after some derivations of a preliminary nature, the response of a tuned circuit will be computed on both the sideband theory and on the Fleming single-frequency theory.

The Time Constant and Circuit Response.

The differential equation of a circuit containing resistance R, inductance L, and capacity C, with an impressed e.m.f. of r.m.s. value e, is

$$L\frac{di}{dt}+Ri+\frac{\int idt}{C}=e. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The solution consists of a steady-state term and a transient term. The latter contains a factor of the form $\epsilon^{-\frac{R}{2L}t}$, whence $\frac{2L}{R}$ is spoken of as the "time constant" of the circuit. In the case where no condenser is present in the circuit the transient term contains the factor $\epsilon^{-\frac{R}{L}t}$, the time constant in this case being $\frac{L}{R}$.

The time constant may be introduced into the expressions for circuit response in the following manner. Let the resonance frequency of the circuit be f_1. If the e.m.f. e be of this frequency the r.m.s. value of the resulting current will be

$$I_1=e/R. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the impressed frequency has some other value $f=\frac{\omega}{2\pi}$ the corresponding current will be

$$I=\frac{e}{R+j\left(L\omega-\frac{1}{C\omega}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The two currents are in the ratio

$$\frac{I}{I_1}=h=\frac{R}{R+j\left(\frac{(LC\omega^2-1)}{C\omega}\right)}. \quad . \quad . \quad . \quad (4)$$

Let a quantity Δ be defined as $\Delta\equiv\frac{f-f_1}{f_1}\equiv\frac{f_2}{f_1}$, so that f_2

represents the amount of detuning considered. The current ratio may now be written

$$h=\frac{1}{1+j\Delta\frac{L\omega_1}{R}\left(\frac{2+\Delta}{1+\Delta}\right)}=\frac{1}{1+j\frac{2\pi Lf_2}{R}\left(\frac{2+\Delta}{1+\Delta}\right)}, \quad . \quad (5)$$

or, in absolute value,

$$h=\frac{1}{\sqrt{1+\left[\frac{2\pi Lf_2}{R}\left(\frac{2+\Delta}{1+\Delta}\right)\right]^2}}. \quad . \quad . \quad . \quad (6)$$

This relation may be expressed in the form

$$\left(\frac{2+\Delta}{1+\Delta}\right)\frac{L}{R}=\frac{\sqrt{1-h^2}}{h}\frac{1}{2\pi f_2}. \quad . \quad . \quad (7)$$

Examination of the quantity on the left discloses the fact that it is identical with the time constant as referred to in the first paragraph for the two cases there considered, for, in the case of a radio-frequency resonant circuit, Δ is a small quantity, and the expression becomes $\frac{2L}{R}$. If the value C is supposed to increase continuously, f_1 will decrease, and eventually a condition will be reached such that Δ is infinite, and the quantity on the left in (7) becomes $\frac{L}{R}$. This is the time constant found for a circuit with infinite series capacitance, viz., containing L and R only.

For the purposes of discussion of circuit response therefore we may take the time constant T of the circuit as

$$T=\left(\frac{2+\Delta}{1+\Delta}\right)\frac{L}{R}. \quad . \quad . \quad . \quad . \quad (8)$$

Relation between Time Constant and Circuit Response.

Let the band width passed by the circuit be defined in the usual manner as the frequency difference f_2 from resonance at which the current $I=0{\cdot}707\ I_1$, or when $h=\frac{1}{\sqrt{2}}$. Inserting this value in (7) we obtain the relation

$$f_2T=\frac{1}{2\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which shows that the product of time constant and band width is a constant.

A simple expression for the response of a resonant circuit to a frequency f_2 cycles different from resonance may be obtained from (6) as follows :—

$$h=\frac{I}{I_1}=\frac{1}{\sqrt{1+(2\pi f_2 T)^2}}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Response of a Resonant Circuit to Modulated e.m.f., Sideband Theory [6].

Let the impressed e.m.f. be of the form

$$e=E_0(1+k\epsilon^{j\omega_2 t})\sin\omega_1 t. \quad . \quad . \quad . \quad . \quad (11)$$

When decomposed into carrier and sidebands the e.m.f. will have the form

$$e=E_0\left[\sin\omega_1 t+j\frac{k}{2}\epsilon^{-j(\omega_1-\omega_2)t}-j\frac{k}{2}\epsilon^{j(\omega_1+\omega_2)t}\right].$$

The instantaneous value of the current will be

$$i=\frac{E_0}{R}\sin\omega_1 t+\frac{jkE_0}{2Z_1}\epsilon^{-j[(\omega_1-\omega_2)t-\phi_1]}-\frac{jkE_0}{2Z_2}\epsilon^{j[(\omega_1+\omega_2)t-\phi_2]}, \quad . \quad . \quad . \quad (12)$$

where

$$Z_1=\sqrt{R^2+\left[L(\omega_1-\omega_2)-\frac{1}{C(\omega_1-\omega_2)}\right]^2}$$

$$Z_2=\sqrt{R^2+\left[L(\omega_1+\omega_2)-\frac{1}{C(\omega_1+\omega_2)}\right]^2}$$

$$\phi_1=\tan^{-1}\frac{R}{L(\omega_1-\omega_2)-\frac{1}{C(\omega_1-\omega_2)}}$$

$$\phi_2=\tan^{-1}\frac{R}{L(\omega_1+\omega_2)-\frac{1}{C(\omega_1+\omega_2)}}.$$

Let the circuit be tuned to the carrier frequency, so that

$$L\omega_1-\frac{1}{C\omega_1}=0. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Then

$$L(\omega_1-\omega_2)-\frac{1}{C(\omega_1-\omega_2)}=L(\omega_1-\omega_2)$$

$$-\left(1+\frac{\omega_2}{\omega_1}\right)L\omega_1=-2L\omega_2$$

and

$$L(\omega_1+\omega_2)-\frac{1}{C(\omega_1+\omega_2)}=L(\omega_1+\omega_2)-\left(1-\frac{\omega_2}{\omega_1}\right)L\omega_1=2L\omega_2.$$

Taking T as $\frac{2L}{R}$,

$$-2L\omega_2=-2\pi f_2TR \quad \text{and} \quad 2L\omega_2=2\pi f_2TR,$$

giving

$$Z_1=R\sqrt{1+(2\pi Tf_2)^2}=Z_2 \text{ and } \phi_1=\tan^{-1}\frac{1}{2\pi Tf_2}=-\phi_2=\phi.$$

With these simplifications the instantaneous current may now be written as

$$i=\frac{E_0}{R}\sin\omega_1 t\left[1+\frac{k\epsilon^{j(\omega_2 t+\phi)}}{\sqrt{1+(2\pi Tf_2)^2}}\right]. \quad . \quad . \quad (14)$$

We are chiefly interested in the mean square current. Computing the mean value of i^2 over both an audio- and a radio-frequency cycle,

$$\bar{i^2}=\frac{E_0^2}{R^2}\frac{\omega_1\omega_2}{4\pi^2}\int_0^{\frac{2\pi}{\omega_1}}\int_0^{\frac{2\pi}{\omega_2}}\sin^2\omega_1 t\left[1+\frac{k\cos(\omega_2 t+\phi)}{\sqrt{1+(2\pi Tf_2)^2}}\right]^2 dt^2$$

$$=i_0^2\left[1+\frac{k^2}{2[1+(2\pi Tf_2)^2]}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where i_0^2 is written for $\frac{E_0^2}{2R^2}$, the mean square of the current with no modulation.

Expression (15), with $2\pi Tf_2=0$, is the one commonly used for computation of the power in a circuit due to excitation by a modulated wave. The term $2\pi Tf_2$ introduces appreciable error if the modulation frequency be high enough, as in television work. The sideband power is appreciably reduced in such cases; the effect is such

that if the percentage modulation in the e.m.f. be k, the percentage modulation of the resulting current is

$$k' = \frac{k}{\sqrt{1+(2\pi \mathrm{T} f_2)^2}}. \quad . \quad . \quad . \quad . \quad (16)$$

A comparison of this expression with (10) shows that the time constant of the system T reduces the percentage modulation in the response at resonance by the same ratio as a sine wave f_2 cycles off resonance is reduced from that at resonance.

Response of a Resonant Circuit to Modulated e.m.f., Single Frequency Theory.

Mathematically, owing to the identity

$$\mathrm{A}(1+k \sin \omega_2 t) \sin \omega_1 t \equiv \mathrm{A} \sin \omega_1 t + \frac{k\mathrm{A}}{2} \cos (\omega_1 - \omega_2)t - k\mathrm{A}/2 \cos (\omega_1+\omega_2)t, \quad . \quad (17)$$

a modulated wave may be considered to be the resultant of three discrete frequencies, which was the point of view adopted in the previous section. These discrete frequencies have objective existence and may be detected by suitable resonating systems, which select the Fourier components. Such systems make use of the right-hand member of the foregoing identity.

It is also possible to adopt the point of view expressed in the left-hand member of the identity, viz., that the modulated wave consists of a single frequency f_1, of which the amplitude varies in accordance with the modulation. It is this dual nature of a modulated wave that has caused confusion in the past.

The situation here is somewhat similar to that in present-day optics. Experiments devised to prove that light consists of wave motion will demonstrate the fact beyond question ; likewise other experiments will prove conclusively that light has a discrete quantum structure. The continuous wave motion and the discrete quanta exist side by side ; either characteristic may be called forth at will by a suitable choice of detecting apparatus.

Similarly with the modulated wave. If one wishes to prove that the sidebands exist one has only to set up a resonating apparatus, such as a wave-meter, which will respond to these components, and they will be found [5]. On account of the very extensive use of resonating

circuits in radio communication, the sideband point of view has been the usual one taken in discussions of modulation; but nevertheless the single-frequency point of view exists; in order to demonstrate its reality we will show that the power increase in a wave due to fluctuations in amplitude resulting from modulation is the same as the power increase due to the presence of sidebands. Let the impressed e.m.f. be of the form given by (11), viz.,

$$e=E_0(1+k\epsilon^{j\omega_2 t})\sin\omega_1 t.$$

The ratio of two consecutive radio-frequency peaks is, considering only the variable portion of the wave,

$$\frac{\epsilon^{j\omega_2(t+t_1)}}{\epsilon^{j\omega_2 t}}=\epsilon^{j\omega_2 t_1}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where $t_1=\frac{1}{f_1}$, the carrier period. The logarithmic decrement of the wave is thus $\delta=j\omega_2 t_1=j2\pi\frac{f_2}{f_1}$ or the damping constant

$$\alpha=\delta f_1=j2\pi f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

We have the expression found by Bjerknes [7] for the mean square of the current I in a tuned circuit of damping constant α_2 when an e.m.f. of maximum value E_0, damping constant α_1, and of the resonant frequency is impressed upon the circuit:

$$\bar{I}^2=\frac{C^2\omega^2}{\tau}\int_0^{\tau}\frac{E_0^2\epsilon^{-2\mu t}\sinh^2 vt}{8L^2C^2\omega^2v^2}\,dt,$$

where $2\mu=\alpha_1+\alpha_2$ and $2v=\alpha_1-\alpha_2$. Upon integration the current squared takes the form

$$\bar{I}^2=\frac{E_0^2[(\alpha_1-\alpha_2)^2-(\alpha_1\epsilon^{-\alpha_2\tau}-\alpha_2\epsilon^{-\alpha_1\tau})^2-\alpha_1\alpha_2(\epsilon^{-\alpha_1\tau}-\epsilon^{-\alpha_2\tau})^2]}{16L^2(\alpha_1-\alpha_2)^2\alpha_1\alpha_2(\alpha_1+\alpha_2)\tau}.$$

If now the incoming wave derives its decrement from modulation at a frequency f_2, so that $\alpha_1=2\pi jf_2$, then τ may be replaced by $\frac{1}{f_2}$, and the current squared upon reduction becomes, upon neglecting terms of higher order than the first in the expansion of $\epsilon^{-\frac{\alpha_2}{f_2}}$,

$$\bar{I}^2=\frac{E_0^2}{2R^2}\frac{1}{1+(2\pi Tf_2)^2}=\frac{E^2}{R^2}\frac{1}{1+(2\pi Tf_2)^2},$$

when E_0 is replaced by its r.m.s. value E. This term represents the power component due only to the variation in amplitude of E ; the steady component, which has no decrement, contributes a term $\overline{I_s}^2 = \frac{E^2}{R^2}$.

The total value of the current squared is thus

$$i^2 = I_s^2 + \frac{k^2}{2} I^2 = i_0^2 \left[1 + \frac{k^2}{2[1+(2\pi T f_2)^2]} \right],$$

which agrees with equation (15) derived on the sideband theory.

Thus it is possible to describe the response of an oscillating system to modulated forces without reference to sidebands, on the assumption that all the energy is concentrated in a single frequency, the carrier.

Conclusion.

It is not the purpose of this paper to enter into an extended discussion of reality. Nevertheless it seems justifiable at this time to at least define, for practical purposes, what we may take as a criterion for the real existence of phenomena.

Sir Ambrose Fleming has suggested that the Cartesian components of a velocity are not real, *i. e.*, that they exist only in the mind of the mathematician, and then solely for convenience in manipulation. The authors would like to point out, however, that if apparatus is mounted upon the moving body in such a way as to indicate velocities in three mutually perpendicular directions the *components may be individually detected.* From a practical standpoint the results of such an experiment may be taken as a criterion for existence—*i. e.*, *if no response is obtained with apparatus specially designed to respond to a certain phenomenon, the phenomenon has no objective existence.*

Thus, when two waves of different frequencies f_1 and f_2 are added together, it might be suspected that the sum of the waves could be replaced by a product term, since

$$\cos 2\pi f_1 t + \cos 2\pi f_2 t \equiv 2 \cos 2\pi \frac{f_1 - f_2}{2} t \cos 2\pi \frac{f_1 + f_2}{2} t.$$

On our definition of reality, however, the sum on the left cannot meaningfully be replaced by the product on the

right, for the frequency $\frac{f_1+f_2}{2}$ cannot be experimentally found when apparatus adjusted to respond to that frequency is used for detection [5] [8]. While the right-hand member exists as a mathematical convenience, it has no " real " existence ; the sum of two waves exists physically only as the sum. A modulated wave *can*, however, exist in two forms given by the identity (17), since both sides of the identity may be experimentally detected ; the sidebands are " real."

This is the criterion tacitly assumed in regard to the dual nature of light. A written discussion of optical interference employing the wave theory, for example, does not imply that, to the authors, quanta are fictitious ; they recognize that quanta may be detected at will by a proper choice of detecting apparatus, and that even in one and the same optical system it is possible for both manifestations to be present at the same time.

Which point of view should be adopted in any particular case depends solely upon convenience. In radio work one deals largely with tuned circuits. These circuits will detect sidebands. It is natural therefore that radio engineers should think in terms of sidebands. On the other hand, there is, for example, the recent proposal of Rebecchini [9] to measure reverberation time of a room with sound-waves modulated a few times per second. Obviously here the single-frequency viewpoint is called for, and the suggested experimental technique is adapted to it and not to the detection of sidebands.

In a general way if the frequency of modulation is low the single-frequency point of view is more convenient ; if the frequency of modulation is high enough to give rise to isolated, easily detectable sidebands, it seems preferable to adopt the sideband concept. In either case the alternative point of view may be adopted whenever it appears desirable to do so.

References.

(1) ' Nature,' cxxv. p. 92 (Jan. 18, 1930).
(2) ' Nature,' cxxv. p. 198 (Feb. 8, 1930).
(3) ' Nature,' cxxv. p. 272 (Feb. 22, 1930).
(4) ' Nature,' cxxv. p. 726 (May 10, 1930).
(5) H. Hazel, Phil. Mag. xix. p. 103 (1935).
(6) A portion of this analysis may be found in Hund, ' High Frequency Measurements,' p. 25 (1933).
(7) Bjerknes, *Ann. d. Physik u. Chemie*, lv. p. 121 (1895).

(8) Such a substitution could be made legitimately on the basis of the reality criterion suggested by Einstein, Podolsky, and Rosen, Phys. Rev. xlvii. p. 777 (1935). Their criterion states :—"If . . . we can predict with certainty (probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity." Here we can predict the frequencies $\frac{1}{2}(f_1-f_2)$ and $\frac{1}{2}(f_1+f_2)$. But if the energy available at these frequencies turns out to be zero, as it does, the frequencies can scarcely be said to have a "real" existence. For this reason the authors prefer the criterion suggested in the text.

(9) Rebecchini, *Ricerche di Ingegneria*, xii. p. 178 (1934).

LXXVII. *On the Theoretical Calculation of Normal Potential of Metals and Henry Constant of Ions.* *By* Iturô Uhara, *Professor at Taihoku High School, Taihoku, Formosa, Japan* *.

ACCORDING to the heat theorem of Nernst the normal potential can be calculated from thermal data for some condensed systems (1). For example, the E.M.F. (E) of a cell,

$$\mathrm{Ag} \mid \text{Saturated solution of AgI} \mid \mathrm{I_2},$$

is expressed by

$$E=E_{0.Ag}-E_{0.I}+2RT \ln C,$$

where $E_{0\,Ag}$, $E_{0.I}$ is normal potential of Ag and I respectively and C is solubility of AgI. The diminution of free energy during the change,

$$Ag+\tfrac{1}{2}I_2=AgI,$$

is expressed by

$$Q_0-\beta T^2-\tfrac{1}{2}\gamma T^3 \ldots$$

if we use a simple algebraic function, where Q_0, β, and γ are determined from the values of the heat of formation of AgI and specific heat of these substances by means of the Nernst theorem. By putting

$$F(E_{0.Ag}-E_{0.I}+2RT \ln C)=Q_0-\beta T^2-\tfrac{1}{2}\gamma T^3 \ldots$$

(where F is Faraday constant) we obtain the relative value of normal potential $(E_{0.Ag}-E_{0.I})$. The cases of

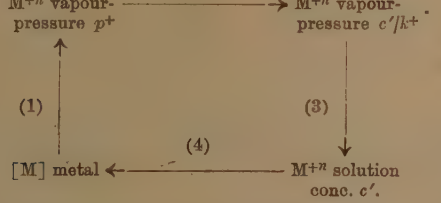
* Communicated by the Author.

AgBr, $PbBr_2$, and PbI_2 were treated similarly, but the range of application of this theory is limited only to the condensed systems as above described.

Bodländer [2] discussed approximately the relations among E_0, solubility, and the heat of formation of compounds instead of the free energy in the above formula. The writer tried the calculation of normal potential of metals from an entirely different standpoint.

Cycle of Evaporation and Electrolysis of Metallic Ion.

The work done in the following reversible cycle is considered. We treat a system so dilute that the influence of ionic atmosphere can be neglected.



(1) 1 gram atom of ion M^{+n} (n is valency of this element) is evaporated from metal [M]. The vapour-pressure of M^{+n} ion in equilibrium with the metal at temperature T is represented by p^+ atm. and electrical work needed in this process by $n\mathrm{F}\phi^+$. As is discussed later the heat of evaporation of ion is divided into two parts, electrical and non-electrical.

Work done : $\mathrm{RT}-n\mathrm{F}\phi^+$.

(2) Pressure of ionic vapour is altered from p^+ atm. to c'/k^+.

c'=concentration of M^{+n} ion in the solution.

k^+ mol/liter. atm.=Henry constant of M^{+n} ion for this solvent. Accordingly c'/k^+ means the vapour-pressure of ion in equilibrium with the solution.

$$\text{Work done : } \mathrm{RT}\ln\frac{p^+k^+}{c'}.$$

(3) Ionic vapour is dissolved in a large quantity of solution.

Work done : $-\mathrm{RT}-n\mathrm{F}\phi_f$,

where $n\mathrm{F}\phi_f$ is the electrical work necessary when 1 mol of ion passes through the surface of the solution.

(4) 1 gram atom of metal is deposited electrolytically from the solution.

Work done : $n\mathrm{E}_0\mathrm{F}+\mathrm{RT}\ln c'$,

E_0 is normal potential of this metal at T in this solvent measured in the direction from solution to metal.

As the sum of work done in the cycle must be zero,

$$+\mathrm{RT}-n\mathrm{F}\phi^+ +\mathrm{RT}\ln\frac{p^+k^+}{c'} -\mathrm{RT}-n\mathrm{F}\phi_f +n\mathrm{E}_0\mathrm{F}+\mathrm{RT}\ln c'=0,$$

$$\therefore \quad \mathrm{E}_0=+\phi_f+\phi^+-\frac{\mathrm{RT}}{n\mathrm{F}}(\ln p^+ +\ln k^+). \quad . \quad . \quad (1)$$

Calculation of $\ln p^+$.

The expression is analogous with Wilson's formula [3] for the vapour-pressure of electron from metal, but the term of specific heat of ion in metal should not be neglected. If we write

L^+ for the heat of evaporation of ion at T. L_0 : at 0° K.

w^+ ,, non-electrical part of L^+. w_0^+ : at 0° K.

$n\mathrm{F}\phi^+$,, electrical part of L^+. $n\mathrm{F}\phi_0^+$: at 0° K.

c_{M}^+ ,, atomic heat of ion in metal.

i^+ ,, (chemical constant of ion)$\times\ln 10$, and put

$$i^+=-3{\cdot}7+\tfrac{3}{2}\ln \mathrm{M}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

M = atomic weight.

As

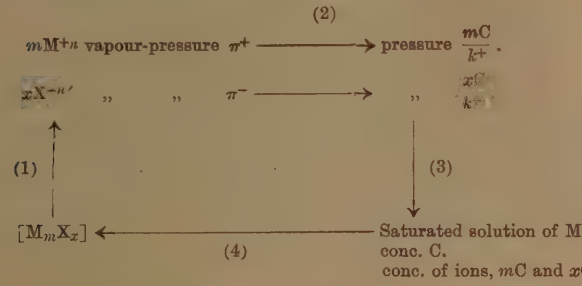
$$\mathrm{L}^+=w^+ +n\mathrm{F}\phi^+ \quad \text{and} \quad w^+=w_0^+ +\tfrac{5}{2}\mathrm{RT}-\int_0^{\mathrm{T}} c_{\mathrm{M}}^+\,d\mathrm{T}, \quad (3)$$

$$\ln p^+=-\frac{\mathrm{L}_0^+}{\mathrm{RT}}-\frac{n\mathrm{F}(\phi^+-\phi_0^+)}{\mathrm{RT}}+\frac{5}{2}\ln \mathrm{T} -\int_0^{\mathrm{T}}\frac{\int_0^{\mathrm{T}} c_{\mathrm{M}}^+\,d\mathrm{T}}{\mathrm{RT}^2}\,d\mathrm{T}+i^+. \quad (4)$$

The Thomson effect is neglected, as this term is usually small compared with other terms.

Calculation of $\ln k^+$.

The work done in the following reversible cycle for a solid compound M_mX_x is calculated, where X being an atom or atomic group forming an anion $X^{-n'}$.



(1) 1 mol of crystal is evaporated into ionic vapour. The vapour-pressure of cation and anion from the crystal is represented by π^+ and π^- atm. respectively, and solubility of M_mX_x in the solvent by C. In case C is small (difficultly soluble compound) and ionization in the solution is perfect, the concentration of ion is mC and xC and vapour-pressure of ion on the saturated solution is $\frac{mC}{k^+}$ and $\frac{xC}{k^-}$ respectively, where k^+ and k^- are Henry constant of ions.

Work done : $+(m+x)RT$.

(2) Pressure of ionic vapour is altered from π^+, π^- to $\frac{mC}{k^+}$, $\frac{xC}{k^-}$ respectively.

$$\text{Work done : } +mRT\ln\frac{\pi^+k^+}{mC}+xRT\ln\frac{\pi^-k^-}{xC}.$$

(3) Vapour of ion is dissolved in a large quantity of saturated solution of M_mX_x.

Work done : $-(m+x)RT$.

(4) 1 mol of crystal of M_mX_x is deposited from a large quantity of saturated solution.

Work done : 0.

As the sum of work done must be zero (the electrical work need not be considered as cation and anion exist in equivalent),

$$+(m+x)\mathrm{RT}+m\mathrm{RT}\ln\frac{\pi^+k^+}{m\mathrm{C}}+x\mathrm{RT}\ln\frac{\pi^-k^-}{x\mathrm{C}}$$

$$-(m+x)\mathrm{RT}+\mathrm{O}=0,$$

$$\therefore\quad m\ln k^+ + x\ln k^- = (m\ln m + x\ln x)$$

$$+(m+x)\ln \mathrm{C}-(m\ln\pi^+ + x\ln\pi^-). \quad (5)$$

We express the lattice energy of $\mathrm{M}_m\mathrm{X}_x$ (heat absorbed when 1 mol of the crystal is evaporated into ionic vapour) at 0° K. by $(m\mathrm{S}_0^+ + x\mathrm{S}_0^-)$, and the atomic heat of ions in the crystal by c^+ and c^-. The lattice energy at 0° K. is determined by Born-Haber's cycle,

$$m\mathrm{S}_0^+ + x\mathrm{S}_0^- \equiv \mathrm{U}_0 = \mathrm{Q}_0 + m(\mathrm{S}_0 + \Sigma\mathrm{I}_0) - xq_0, \quad . \quad . \quad (6)$$

where Q_0 = the heat of formation of crystal at 0° K.

S_0 = the heat of sublimation of metal at 0° K.

$\Sigma\mathrm{I}_0$ = the heat of ionization of atom. at 0° K.,

$$e.\,g.,\ \mathrm{Mg}\xrightarrow{\mathrm{I}_1}\mathrm{Mg}^+\xrightarrow{\mathrm{I}_2}\mathrm{Mg}^{++},\quad \Sigma\mathrm{I}_0=\mathrm{I}_1+\mathrm{I}_2.$$

q_0 = the heat of formation of anion at 0° K.,

$$e.\,g.,\ \tfrac{1}{2}[\mathrm{I}_2]\xrightarrow{q_1}\tfrac{1}{2}\mathrm{I}_2\xrightarrow{q_2}\mathrm{I}\xrightarrow{q_3}\mathrm{I}^-,$$

$$[\mathrm{C}]+\tfrac{3}{2}\mathrm{O}_2\xrightarrow{q_1}\mathrm{CO}_3\xrightarrow{q_2}\mathrm{CO}_3^{--},\quad q_0=\Sigma q.$$

$$m\ln\pi^+ + x\ln\pi^-$$

$$= m\left(-\frac{\mathrm{S}_0^+}{\mathrm{RT}}+\frac{5}{2}\ln\mathrm{T}-\int_0^{\mathrm{T}}\frac{\int_0^{\mathrm{T}}c^+\,d\mathrm{T}}{\mathrm{RT}^2}d\mathrm{T}+i^+\right)$$

$$+x\left(-\frac{\mathrm{S}_0^-}{\mathrm{RT}}+\frac{5}{2}\ln\mathrm{T}-\int_0^{\mathrm{T}}\frac{\int_0^{\mathrm{T}}c^-\,d\mathrm{T}}{\mathrm{RT}^2}d\mathrm{T}+i^-\right). \quad (5')$$

Calcutation of Normal Potential.

We put

$$mc^+ + xc^- = c_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where c_c is molecular heat of M_mX_r, and extend Schottky's relation [4] to n-valent cation,

$$L_0^+ + nL_{e0} = S_0 + \Sigma I_0, \quad \ldots\ldots \quad (8)$$

where L_{e0} is heat of evaporation of electron from metal at 0° K. Introducing (3)∼(8) in (1), we obtain the following expression:

$$E_0 = \left(+\phi_f + 2\phi^+ - \phi_0^+ - \frac{L_{e0}}{F}\right) - \frac{RT}{mnF}\left\{(m \ln m + x \ln x) + (m+x)\ln C + \frac{Q_0}{RT} + \int_0^T \frac{\int_0^T c_c dT}{RT^2} dT - m\int_0^T \frac{\int_0^T c_M^+ dT}{RT^2} dT\right\} + \frac{RT}{nF}\cdot\frac{x}{m}\left\{\frac{5}{2}\ln T + i^- + \frac{q_0}{RT} + \ln k^-\right\}. \quad \ldots \quad (9)$$

Discussion on Formula (9)

ϕ_f, potential difference at the surface of solution, is considered to be practically equal to that of pure solvent, as we deal with very dilute solution (ion is negatively adsorbed at the surface of aqueous solution). Even in case of 1∼2 mol solution it is observed that surface potential is exclusively determined by anion [5].

If we divide the heat of evaporation of electron from metal into electrical and non-electrical parts,

$$L_e = F\phi_e + w_e$$

and $$L_{e0} = F\phi_{e0} + w_{e0} \quad \text{at } 0° \text{ K}.$$

According to Richardson [6]

$$w_{e0} = 0, \quad \therefore\ L_{e0} = F\phi_{e0}.$$

The electrical part of the heat of evaporation, being the work against the electrostatical force caused by induction when the charge passes through the surface, is independent of the sign of the electric charge,

$$\therefore\ \phi^+ = \phi_e, \qquad \phi_0^+ = \phi_{e0}.$$

Dushman's formula [9], which expresses the experimental results on thermionic current very well, is obtained if we assume

$$\phi_e = \phi_{e0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\therefore \left(+\phi_f + 2\phi^+ - \phi_0^+ - \frac{L_{e0}}{F} \right) = +\phi_f = \text{constant depending only on the solvent (and anion if the solution is not dilute).} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The heat of formation of M_mX_x at T is expressed as follows:—

$$Q = Q_0 + m\int_0^T c_M\,dT + x\int_0^T \Sigma\,(\text{atomic heat of element forming anion})\,dT - \int_0^T c_e\,dT. \quad . \quad . \quad . \quad (12)$$

The terms

$$\frac{RT}{nF}\cdot\frac{x}{m}\left\{\frac{5}{2}\ln T + \ldots\right\} \text{ in (9)} \quad \text{and}$$

$$\frac{1}{mnF}\cdot x\int_0^T \Sigma\,(\text{atomic heat of element forming anion})\,dT,$$

depend only on anion and solvent. As the specific heat of electron in metal is negligible so long as T is not very high, we can write

$$c_M^+ \fallingdotseq c_M.$$

$$\therefore \quad E_0 = -\frac{RT}{mnF}\left\{ (m\ln m + x\ln x) + (m+x)\ln C + \frac{Q}{RT} + \int_0^T \frac{c_e}{RT}\,dT - m\int_0^T \frac{c_M}{RT}\,dT \right\}$$

$$+ (\text{constant depending on anion and solvent only}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If two difficultly soluble compounds of two metals of common anion in which solution ionization is complete, are taken, the relative value of normal potential of these two metals is calculated from values of heat of formation, solubility, and entropy [8] of these compounds and metals. For instance, 0·95 volt is obtained from data of AgCl and $PbCl_2$ and 0·97 volt from AgI and PbI_2 for $(E_{0.Ag} - E_{0.Pb})$ at 25°. According to Landolt, 'Tabellen,' 0·94, and Int. Crit. Table, 0·92 volt are given for $(E_{0.Ag} - E_{0.Pb})$.

As the numerical value of the entropy term in (13) is usually unknown, we assume approximately that

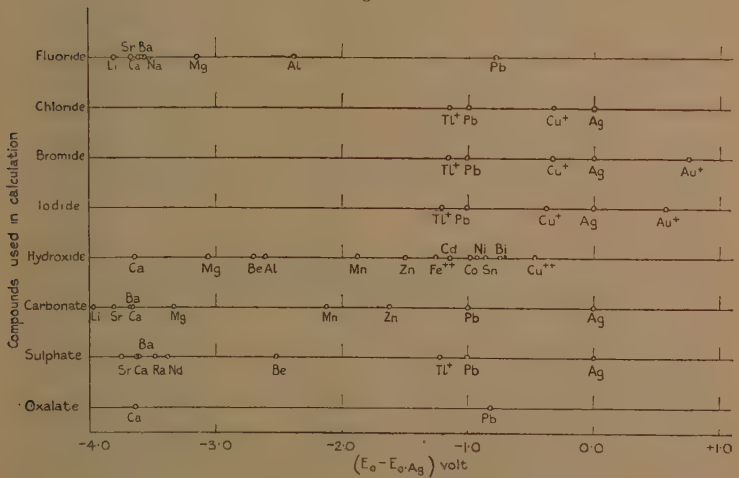
$$\frac{1}{mn}\left\{\int_0^T \frac{c_e}{RT}dT - m\int_0^T \frac{c_M}{RT}dT\right\}$$

is constant for all metals if the anion is common. This assumption is true if Neumann-Kopp's rule is valid rigorously, and nearly so in the case of like metals.

Value of $\frac{1}{mn}\left\{\int_0^T \frac{c_e}{RT}dT - m\int_0^T \frac{c_M}{RT}dT\right\}$ (25°).

$PbCl_2$.	AgCl.	NaCl.	KCl.	PbI_2.	AgI.	CaO.	ZnO.
4·6	6·5	2·4	2·4	6·6	8·3	−0·3	+0·2

Fig. 1.



Then

$$E_0 = -\frac{RT}{mnF}\left\{(m \ln m + x \ln x) + (m+x)\ln C + \frac{Q}{RT}\right\}$$

$$+(\text{term depending on anion and solvent only}), \quad (14)$$

or written in a simple form,

$$E_0 = -B + (\text{term of anion and solvent}). \quad (14')$$

TABLE I.—Q, k. cal. C, mol/l. B, volt at 25°. Solvent water.

	Q.	C.	B.		Q.	C.	B.		Q.	C.	B.
AgCl	30·6	$1{\cdot}42\times10^{-5}$	0·75	BaF_2	281·6	$9{\cdot}2\times10^{-3}$	5·94	Ag_2CO_3	120·4	$1{\cdot}2\times10^{-4}$	2·28
CuCl	32·9	1×10^{-3}	1·07	SrF_2	289·3	$9{\cdot}4\times10^{-4}$	6·02	$PbCO_3$	169·5	$1{\cdot}8\times10^{-7}$	3·28
$PbCl_2$	85·7	$3{\cdot}88\times10^{-2}$	1·75	LiF	144·7	$6{\cdot}2\times10^{-2}$	6·15	$ZnCO_3$	194·2	$4{\cdot}6\times10^{-5}$	3·90
TlCl	48·5	$1{\cdot}6\times10^{-2}$	1·90	PbC_2O_4	205·3	$6{\cdot}08\times10^{-6}$	4·14	$MnCO_3$	210·8	$1{\cdot}1\times10^{-3}$	4·40
AuBr	−0·1	—	−0·44	CaC_2O_4	332·9	$*4{\cdot}76\times10^{-5}$	6·96	$MgCO_3$	266·6	$1{\cdot}3\times10^{-3}$	5·61
AgBr	24·0	$7{\cdot}15\times10^{-7}$	0·31	$Cu(OH)_2$	107·8	1×10^{-6}	1·81	$CaCO_3$	284·5	$1{\cdot}4\times10^{-4}$	5·94
CuBr	25·0	2×10^{-4}	0·64	$Bi(OH)_3$	171·5	6×10^{-6}	2·09	$BaCO_3$	285·6	$1{\cdot}8\times10^{-4}$	5·96
$PbBr_2$	66·4	$2{\cdot}6\times10^{-2}$	1·32	$Sn(OH)_2$	136·4	$2{\cdot}3\times10^{-9}$	2·20	$SrCO_3$	292·1	$6{\cdot}8\times10^{-5}$	6·09
TlBr	41·3	$2{\cdot}0\times10^{-3}$	1·47	$Ni(OH)_2$	129·1	$0{\cdot}6\times10^{-6}$	2·27	Li_2CO_3	291·2	$1{\cdot}71\times10^{-1}$	6·25
AuI	−5·5	—	−0·90	$Co(OH)_2$	131·7	$0{\cdot}7\times10^{-6}$	2·32	Ag_2SO_4	165·2	$2{\cdot}6\times10^{-2}$	3·47
AgI	14·35	$1{\cdot}0\times10^{-8}$	−0·33	$Cd(OH)_2$	131·9	4×10^{-5}	2·48	$PbSO_4$	216·8	$1{\cdot}45\times10^{-4}$	4·48
CuI	16·3	$2{\cdot}6\times10^{-6}$	0·05	$Fe(OH)_2$	136·6	$6{\cdot}7\times10^{-5}$	2·60	Tl_2SO_4	219·1	$8{\cdot}11\times10^{-2}$	4·69
PbI_2	41·9	$1{\cdot}65\times10^{-3}$	0·68	$Zn(OH)_2$	153·8	2×10^{-6}	2·84	$BeSO_4$	277·5	$*4{\cdot}1\times10^{-1}$	5·99
TlI	30·2	$2{\cdot}45\times10^{-4}$	0·88	$Mn(OH)_2$	163·1	$1{\cdot}2\times10^{-4}$	3·22	$Nd_2(SO_4)_3$	928·2	$*1{\cdot}15\times10^{-1}$	6·85
PbF_2	155·9	$2{\cdot}7\times10^{-3}$	3·13	$Al(OH)_3$	297·0	$1{\cdot}9\times10^{-5}$	3·95	$RaSO_4$	340	$6{\cdot}5\times10^{-8}$	6·95
AlF_3	331·5	$*6{\cdot}7\times10^{-2}$	4·73	$Be(OH)_2$	206·4	$0{\cdot}8\times10^{-5}$	4·04	$CaSO_4$	332·5	$*1{\cdot}5\times10^{-2}$	7·10
MgF_2	264·3	$1{\cdot}36\times10^{-3}$	5·50	$Mg(OH)_2$	217·3	2×10^{-4}	4·40	$BaSO_4$	340·2	$1{\cdot}1\times10^{-5}$	7·09
NaF	136·6	$1{\cdot}03\times10^{0}$	5·91	$Ca(OH)_2$	236·0	$2{\cdot}12\times10^{-2}$	4·98	$SrSO_4$	341·8	$6{\cdot}3\times10^{-4}$	7·22
CaF_2	289·4	$2{\cdot}15\times10^{-4}$	5·96								

Value of C with (*) is that of hydrated compound, but error is usually small.
Solubility of AuBr, AuI is assumed to be near that of CuBr, CuI respectively. Q of $RaSO_4$ is assumed to be 340 k. cal.

The relative value of normal potential of two metals is expressed in terms of two compounds of common anion,

$$E_{01}-E_{02}=B_2-B_1 \quad . \quad . \quad . \quad . \quad . \quad (15)$$

In case ionization in the solution is incomplete, these formulæ cannot be used, *e. g.*, mercury salts.

TABLE II.

	$(E_0.Ag-E_0)$ mean.	E_0: calc. putting $E_{0.Ag}=+0{\cdot}80$ volt.	Landolt-Tab. ($E_{0.H}=0{\cdot}00$ volt).	Int. Crit. Tab.	Other literatures.
Au^{+}	$-0{\cdot}67$	$+1{\cdot}47$	$+1{\cdot}5$		
Ag^{+}	$0{\cdot}00$	$+0{\cdot}80$	$+0{\cdot}81$	$+0{\cdot}80$	
Cu^{+}	$0{\cdot}34$	$+0{\cdot}46$	$+0{\cdot}52$		
Cu^{++}	$0{\cdot}48$	$+0{\cdot}32$	$+0{\cdot}34$	$+0{\cdot}34$	
Bi^{+++}	$0{\cdot}75$	$+0{\cdot}05$	$+0{\cdot}2$		
Pb^{++}	$0{\cdot}95$	$-0{\cdot}15$	$-0{\cdot}13$	$-0{\cdot}12$	
Sn^{++}	$0{\cdot}87$	$-0{\cdot}07$	$-0{\cdot}14$	$-0{\cdot}14$	
Ni^{++}	$0{\cdot}94$	$-0{\cdot}14$	$-0{\cdot}25$	$-0{\cdot}23$	
Co^{++}	$0{\cdot}99$	$-0{\cdot}19$	$-0{\cdot}25$		
Tl^{+}	$1{\cdot}18$	$-0{\cdot}38$	$-0{\cdot}34$		
Cd^{++}	$1{\cdot}15$	$-0{\cdot}35$	$-0{\cdot}40$	$-0{\cdot}40$	
Fe^{++}	$1{\cdot}27$	$-0{\cdot}47$	$-0{\cdot}44$	$-0{\cdot}44$	
Zn^{++}	$1{\cdot}57$	$-0{\cdot}77$	$-0{\cdot}76$	$-0{\cdot}76$	
Mn^{++}	$2{\cdot}01$	$-1{\cdot}21$	$-1{\cdot}1$		
Al^{+++}	$2{\cdot}50$	$-1{\cdot}70$			$-1{\cdot}7$
Be^{++}	$2{\cdot}52$	$-1{\cdot}72$	$-1{\cdot}7$		
Mg^{++}	$3{\cdot}19$	$-2{\cdot}39$	$-2{\cdot}35$		$-2{\cdot}40$
Nd^{+++}	$3{\cdot}38$	$-2{\cdot}58$			
Na^{+}	$3{\cdot}57$	$-2{\cdot}77$	$-2{\cdot}71$	$-2{\cdot}71$	
Ra^{++}	$3{\cdot}48$	$-2{\cdot}68$			
Ca^{++}	$3{\cdot}64$	$-2{\cdot}84$	$-2{\cdot}76$	$-2{\cdot}76$	$-2{\cdot}87$
Ba^{++}	$3{\cdot}64$	$-2{\cdot}84$			$-2{\cdot}90$
Sr^{++}	$3{\cdot}75$	$-2{\cdot}95$			$-2{\cdot}92$
Li^{+}	$3{\cdot}88$	$-3{\cdot}08$	$-3{\cdot}02$	$-2{\cdot}96$	

The relative values of normal potential calculated are shown in fig. 1, in which it is seen that anion is approximately independent on them.

E_0 value calculated only from the datum of hydroxide of which C is not exact, and complete ionization is doubtful; Bi, etc., deviates considerably from that given in literature.

Normal Potential in Non-aqueous Solvents.

If the ionization is complete (13) and (14) apply to non-aqueous solvents.

TABLE III.—25°.

	Solvent=CH_3OH.			C_2H_5OH.		
	Q k. cal.	C mol/1.	B volt.	Q k. cal.	C mol/1.	B volt.
$AgNO_3$	28·7	$1·73\times10^{-3}$	0·93	28·7	0·167	1·15
$Pb(NO_3)_2$ ..	108·1	$3·28\times10^{-4}$	2·05	108·1	$0·955\times10^{-3}$	2·10
$CuSO_4$	180·5	$5·30\times10^{-4}$	3·72			
$ZnSO_4$	230·1	$3·18\times10^{-4}$	4·79			
$(E_{0.Ag}-E_{0.Pb})$	calc. 1·12.			$(E_{0.Ag}-E_{0.Pb})$	calc. 0·95.	
	obs.(*) 0·98.				obs. 0·95.	
$(E_{0.Cu}-F_{0Zn})$	calc. 1·07.					
	obs. 1·11.					

Q, which is independent of the solvent, has the greatest influence on E_0; accordingly description by Sackur [10] that "Normal potential depends on the solvent and temperature but the order of magnitude of which is independent of them," is true in most cases.

Calculation of Q.

If $(E_{01}-E_{02})$ and B_1 are known, B_2 is determined, therefore the heat of formation Q can be calculated in case solubility C is known. For example, C of Tl_2CO_3 and SnI_2 is $1·2\times10^{-1}$ and $2·9\times10^{-2}$ mol/1, then we obtain 162 and 34 k. cal. respectively for Q. Though it seems possible to calculate C from Q, the error is too large for quantitative purposes, as this term is usually small.

Another Expression of E_0.

(13) is differentiated with respect to T,

$$\frac{dE_0}{dT}=-\frac{R}{mnF}\left\{(m\ln m+x\ln x)+(m+x)\left(\ln C+T\frac{d\ln C}{dT}\right)+\int_0^T\frac{c_c}{RT}dT-m\int_0^T\frac{c_M}{RT}dT\right\}+(\text{term of anion and solvent}).$$

The energy evolved when 1 gram atom of ion in solution is deposited as metal is expressed by u, then, according to Gibbs-Helmholtz's equation,

$$E_0 = \frac{u}{nF} + T\frac{dE_0}{dT},$$

$$\therefore \quad E_0 = \frac{u}{nF} - \frac{RT}{mnF}\left\{(m \ln m + x \ln x) + (m+x)\left(\ln C + T\frac{d \ln C}{dT}\right) + \int_0^T \frac{c_c}{RT}dT - m\int_0^T \frac{c_M}{RT}dT\right\}$$

$$+(\text{constant depending on anion and solvent only}). \quad \ldots \quad (16)$$

If the same assumption for the entropy term used in deriving (14) is adopted,

$$E_0 = \frac{u}{nF} - \frac{RT}{mnF}\left\{(m \ln m + x \ln x) + (m+x)\left(\ln C + T\frac{d \ln C}{dT}\right)\right\}$$

$$+(\text{constant depending on anion and solvent}), \quad (17)$$

or in a simple form,

$$E_0 = \frac{u}{nF} - G + (\text{constant depending on anion and solvent}). \quad \ldots \quad (17')$$

For two difficultly soluble compounds of common anion,

$$E_{01} - E_{02} = \frac{u_1}{n_1F} - \frac{u_2}{n_2F} - G_1 + G_2. \quad \ldots \quad (18)$$

Determination of u.

The relative value of u is easily determined from the heat of solution of the metal in dilute aqueous solution of strong acid with evolution of H_2. Though the absolute value is obtained if u of H_2 is put zero [11], it is unnecessary for our present purpose. In the case of noble metal it is calculated from the values of heat of formation of

oxide, heat of solution of oxide in dilute strong acid, and heat of formation of water.

TABLE IV.

u k. cal. and u/nF volt at 25°.

	u.	u/nF.		u.	u/nF.
Li	− 66·7	−2·89	Ba	−128·5	−2·78
Na	− 58·1	−2·52	Zn	− 36·3	−0·79
K	− 60·9	−2·64	Cd	− 17·2	−0·37
Rb	− 61·0	−2·64	Hg^{++}	+ 40 ?	+0·87 ?
Cs	− 62·2	−2·70	Al	−127·0	−1·84
Cu^{++}	+ 15	+0·33	Ce^{+++}	−165	−2·38
Ag	+ 25·5	+1·10	Sn^{++}	− 2·5	−0·05
Tl^{+}	− 1·0	−0·04	Pb	− 1	−0·02
Be	− 84·9	−1·84	Fe^{++}	− 20·8	−0·45
Mg	−110·2	−2·39	Co	− 15	−0·33
Ca	−129·8	−2·82	Ni	− 15	−0·33
Sr	−130·2	−2·82	Mn^{++}	− 49·8	−1·08

$$\text{Value of } T\frac{d\ln C}{dT}.$$

The equation

$$T\frac{d\ln C}{dT} = \frac{e}{RT}$$

(e is the last heat of solution),

is not applicable, as the value of e is not exact, especially in cases of difficultly soluble compounds, for it is determined as the difference of large amount of heat. In our cases it is therefore calculated directly from solubility data.

Results of calculation from (17) and (18) are given in fig. 2 and Table VI.

Numerical Comparison of Henry Constant of Ions.

From (1),

$$\ln k^{+} = (+\phi_f + \phi^{+} - E_0)\frac{nF}{RT} - \ln p^{+}.$$

TABLE V.—C mol/1, $T\frac{d \ln C}{dT}$, G volt at 25°. Solvent=water.

	C.	$T\frac{d \ln C}{dT}$.	G.
LiF	$6{\cdot}2\times10^{-2}$	+1·8	−0·05
BaF_2	$9{\cdot}2\times10^{-3}$	+0·3	−0·15
SrF_2	$9{\cdot}4\times10^{-4}$	+0·5	−0·23
CaF_2	$2{\cdot}15\times10^{-4}$	+2·1	−0·31
MgF_2	$1{\cdot}36\times10^{-3}$	−1·1	−0·28
PbF_2	$2{\cdot}7\times10^{-3}$	+2·1	−0·17
TlCl	$1{\cdot}6\times10^{-2}$	+10	+0·31
$PbCl_2$	$3{\cdot}88\times10^{-2}$	+5·5	+0·10
TlBr	$2{\cdot}0\times10^{-3}$	+11·5	+0·27
$PbBr_2$	$2{\cdot}6\times10^{-2}$	+7·7	+0·18
SnI_2	$2{\cdot}9\times10^{-2}$	+4·8	+0·06
PbI_2	$1{\cdot}65\times10^{-3}$	+8·0	+0·08
TlI	$2{\cdot}45\times10^{-4}$	+15·4	+0·37
$Ca(OH)_2$	$2{\cdot}12\times10^{-2}$	−2·8	−0·25
$Mg(OH)_2$	2×10^{-4}	0*	−0·31
$BaSO_4$	$1{\cdot}1\times10^{-5}$	+3·5	−0·21
$SrSO_4$	$6{\cdot}3\times10^{-4}$	0·0	−0·19
$CaSO_4$	†$1{\cdot}5\times10^{-2}$	−7·1	−0·29
Tl_2SO_4	$8{\cdot}11\times10^{-2}$	+6·1	+0·17
$PbSO_4$	$1{\cdot}45\times10^{-4}$	+1·9	−0·18
Ag_2SO_4	$2{\cdot}6\times10^{-2}$	+3·6	+0·01
Li_2CO_3	$1{\cdot}71\times10^{-1}$	−2·8	−0·16
$BaCO_3$	$1{\cdot}8\times10^{-4}$	+4·3	−0·11
$CaCO_3$	$1{\cdot}4\times10^{-4}$	+0·6	−0·21
Tl_2CO_3	$1{\cdot}2\times10^{-1}$	+7	+0·04
$CsClO_3$	$3{\cdot}28\times10^{-1}$	+12·2	+0·56
$RbClO_3$	$4{\cdot}85\times10^{-1}$	+11·1	+0·53
$KClO_3$	$6{\cdot}44\times10^{-1}$	+ 9·5	+0·47
$CsClO_4$	$7{\cdot}86\times10^{-2}$	+11·4	+0·46
$RbClO_4$	$7{\cdot}04\times10^{-2}$	+13·8	+0·57
$KClO_4$	$1{\cdot}60\times10^{-1}$	+12·2	+0·53
$Pb(IO_3)_2$	$3{\cdot}95\times10^{-5}$	+ 8·4	−0·05
$AgIO_3$	$1{\cdot}77\times10^{-4}$	+11·7	+0·16
$BaCrO_4$	$1{\cdot}62\times10^{-5}$	+ 5·9	−0·13
Ag_2CrO_4	$9{\cdot}65\times10^{-5}$	+ 8·4	−0·02
$CaCrO_4$	$1{\cdot}31\times10^{-1}$	− 4·5	−0·17
SrC_2O_4	$2{\cdot}95\times10^{-4}$	+ 4·8	−0·10
PbC_2O_4	$6{\cdot}08\times10^{-6}$	+11·4	−0·02
$Ag_2C_2O_4$	$1{\cdot}35\times10^{-4}$	+ 7·7	0·00
$CH.OH.CO_2$ / $CH_2.CO_2$ Ca	$4{\cdot}65\times10^{-2}$	− 1·7	−0·12
$CH.OH.CO_2$ / $CH_2.CO_2$ Ba	$3{\cdot}34\times10^{-2}$	0·0	−0·06
$CHO.H.CO_2$ / $CH.OH.CO_2$ Ba	$1{\cdot}04\times10^{-3}$	+ 3·6	−0·09
$CH.OH.CO_2$ / $CH.OH.CO_2$ Pb	$3{\cdot}03\times10^{-4}$	+ 3·4	−0·13

* Thermal datum.

† C of $CaSO_4.2H_2O$ is used. $T\frac{d \ln C}{dT}$ of $CaSO_4.2H_2O = +0{\cdot}7$ (heat of hydration of $CaSO_4$)/RT = +7·8.

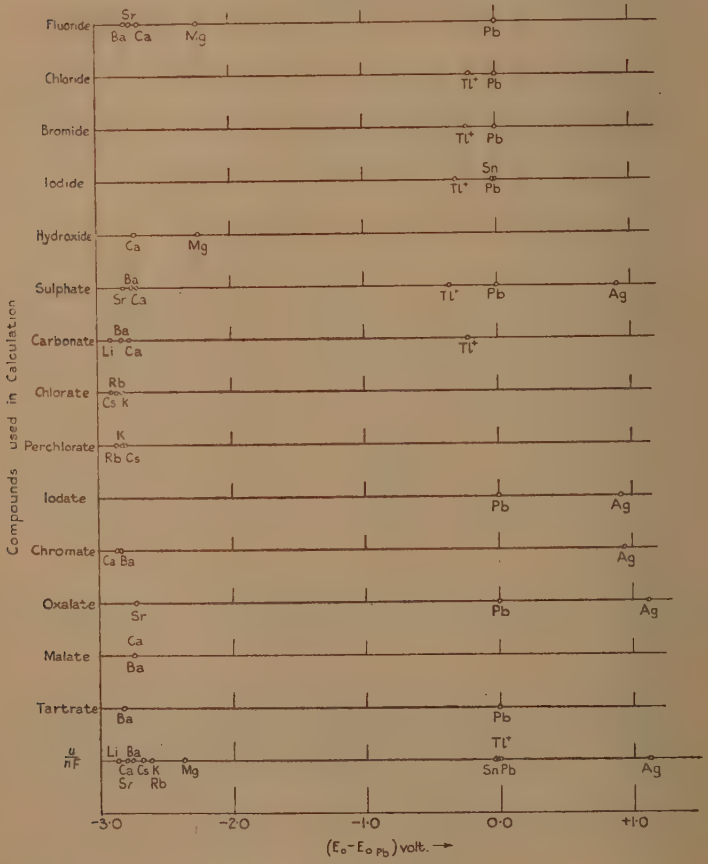
$\therefore$ $T\frac{d \ln C}{dT}$ of $CaSO_4 = -7{\cdot}1$.

Introducing (2), (4), (8), (11), and the following relation,

$$S = S_0 + \tfrac{5}{2} RT - \int_0^T c_M \, dT,$$

and putting $c_M^{\pm} \doteq c_M$,

Fig. 2.



the following expression is obtained:

$$\ln k^{+} = -E_0 \frac{nF}{RT} + \frac{\Sigma I_0}{RT} + \frac{S}{RT} + \int_0^T \frac{c_M}{RT} dT - \tfrac{5}{2} \ln T - \tfrac{3}{2} \ln M + 1{\cdot}2 + n\phi_f. \quad (19)$$

TABLE VI.

	$(E_{0.Pb}-E_0)$ mean.	E_0 : calc. putting $E_{0.Pb}=-0{\cdot}12$ volt.	Landolt-Tab. ($E_{0.H}=0{\cdot}00$ volt.)	Int. Crit. Tab. ($E_{0.H}=0{\cdot}00$ volt.)	Other sources.
Li	+2·93	−3·05	−3·02	−2·96	
K^+	+2·79	−2·91	−2·92	−2·92	
Rb^+.....	+2·84	−2·96	−2·92		
Cs^+	+2·86	−2·98			
Ba^{++} ...	+2·78	−2·90			−2·90
Sr^{++}.....	+2·76	−2·88			−2·92
Ca^{++} ...	+2·72	−2·84	−2·76	−2·76	−2·87
Mg^{++} ...	+2·24	−2·36	−2·35		
Tl^+	+0·22	−0·34	−0·34		
Sn^{++} ...	+0·01	−0·13	−0·14	−0·14	
Pb^{++} ...	0·00	−0·12	−0·12	−0·12	
Ag^+.....	−0·92	+0·80	+0·81	+0·80	

TABLE VII.

Value of $\int_0^{1T} \frac{c_M}{RT} dT$ at 25° (those in brackets are assumed values).

Li.	Na.	K.	Rb.	Cs.	Cu.	Ag.	Tl.	Ca.	Sr.	Ba.
(5)	6·1	7·6	(7)	(7)	3·9	5·0	(5)	5·0	(6)	(6)

Mg.	Sn.	Pb.	Mn.	Fe.	Co.	Ni.	Zn.	Cd.	Hg.	Al.
(5)	(7)	7·8	(5)	(4)	(5)	(6)	5·0	6·2	9·1	3·3

S : given by Sherman [12].

TABLE VIII.

$\{(\ln k^+) - n(\ln k^+ \text{ of } Na^+)\}$ at 25°. Solvent = water.

Li^+.	Na^+.	K^+.	Rb^+.	Cs^+.	Cu^+.	Ag^+.	Tl^+.
+40	0	−30	−41	−51	+64	23	−27

Ca^{++}.	Sr^{++}.	Ba^{++}.	Mg^{++}.	Sn^{++}.	Pb^{++}.	Mn^{++}.	Fe^{++}.
+312	257	220	436	307	272	421	447

Co^{++}.	Ni^{++}.	Cu^{++}.	Zn^{++}.	Cd^{++}.	Hg^{++}.	Al^{+++}.
+456	487	524	482	392	404 ?	+1326

The Heat of Hydration of Ion.

This is the heat evolved when 1 gram atom of ion in vacuum is dissolved in water and is designated by W^+ and W^-. According to Fajans and Schwartz [13],

$$mW^+ + xW^- = U + L. \quad . \quad . \quad . \quad . \quad (20)$$

U : lattice energy of crystal of M_mX_x,
L : first heat of solution of this crystal.

TABLE IX.

$\{W^+ - n(W^+ \text{ of } Na^+)\}$ k. cal./mol at 25°.

Those of alkali metal ions are obtained from data given by Meyer and Helmholtz [14], the others are calculated from (21).

	Li^+.	Na^+.	K^+.	Rb^+.	Cs^+.
	+25	0	−20	−27	−34

From	Cu^+.	Ag^+.	Tl^+.	Cu^{++}.	Mg^{++}.	Ca^{++}.	Sr^{++}.	Ba^{++}.
Chloride...	47	—	−12	315	264	189	154	126
Nitrate ...	—	14	−16	314	263	181	151	118
Sulphate...	—	15	−14	311	264	181	151	121
Mean ...	47	15	−14	313	264	183	152	122

From	Zn^{++}.	Cd^{++}.	Hg^{++}.	Sn^{++}.	Pb^{++}.	Mn^{++}.	Fe^{++}.	Co^{++}.	Ni^{++}.	Al^{++}.
Chloride...	294	238	—	184	172	255	274	277	294	796
Nitrate ...	291	237	—	—	164	254	274	—	—	794
Sulphate...	292	237	239 ?	—	163	255	—	—	—	795
Mean ...	292	237	239 ?	184	166	255	274	277	294	795

The result of (6) holds at T,

$$U = Q + m(S + \Sigma I) - xq,$$

$$\therefore \quad W^+ = \frac{Q+L}{m} + (S + \Sigma I) - \frac{x}{m}(q + W^-) = \frac{Q+L}{m} + (S + \Sigma I)$$

$$- n \text{ (constant depending on anion).} \quad (21)$$

Though we put

$$\Sigma I = \Sigma I_0 + \tfrac{5}{2} nRT,$$

the temperature term vanishes when the difference is taken as in Table IX.

As is shown in figs. 3 and 4, the parallelism between W^+ and k^+ is very remarkable, which is estimated from (19) and (21), too.

Fig. 3.

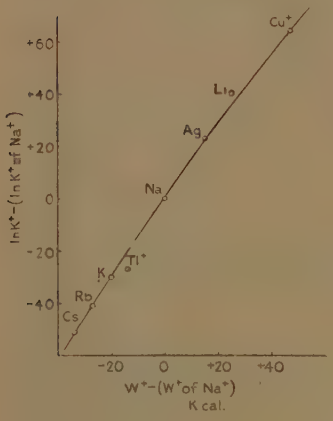
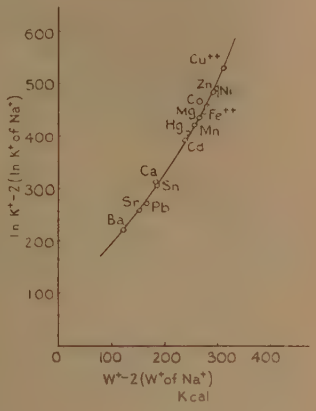


Fig. 4.



The relation among these quantities and ionic radius and other properties will be discussed in the next paper.

The author wishes to express his heartfelt thanks to Prof. M. Katayama for his kind guidance.

Summary.

(1) Expressions for normal potential of metals in solvents were deduced by means of thermodynamical cycle.

(2) Calculated values from the heat of formation and solubility of difficultly soluble compounds of metals were compared with those given in literatures.

(3) Parallelism between Henry constant and heat of hydration of ion was observed.

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Taihoku High School,
Taihoku, Formosa, Japan.
21 July, 1935.

LXXVIII. *Concealed Systems and Relativity.* *By* Robert Cameron Colwell, *Professor of Physics, West Virginia University* *.

In the Newtonian mechanics of material bodies as distinct from that of relativity, each material body is given a definite mass m and its motion determined from Newton's laws. These laws are found by experiment, and the real cause of the motion is unknown. The system which underlies all Newtonian mechanics is a concealed system existing in space and interacting with matter in a manner not yet understood. Nevertheless, despite our ignorance of the concealed system, we can still solve the problems of mechanics by using the experimental law that the rate of change of momentum is proportional to the impressed force. In just the same way we may find the laws of motion of any system without knowing the nature of the concealed system which is the cause of the laws. This is made clear by Maxwell's analogy †. Suppose that a certain dynamical system is presented to us for examination. Its parts are all concealed from us except that there are certain ropes hanging down from the enclosed machinery (for simplicity I shall take two ropes). If on pulling one rope down a distance of a feet the other rope rises b feet, we have found out all we need to know about the mechanism in so far as it determines the motion of the ropes. We can, therefore, disregard the nature of the concealed mechanism,

* Communicated by the Author.
† Maxwell, 'Scientific Papers,' vol. ii. p. 783.

but we are certainly not justified in stating that the mechanism is non-existent.

These principles are applied to practical problems in mechanics by a generalization into Lagrange's equations,

$$\frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{\theta}_n} - \frac{\partial \mathrm{L}}{\partial \theta_n} = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which show that it is possible to find the state of the external system if the kinetic and potential energies are known without regard to the nature of the mechanism involved.

Suppose that we build a mechanism of cog-wheels with two ropes hanging from it, designed so that if one rope is pulled down a feet the other rises b feet. Suppose, however, that, after the mechanism is set up, we find that on pulling down the first rope a feet the second rope rises $(b+c)$ feet. If we are absolutely certain of the ratio of the cog-wheels, we must assume that the ropes are changing their lengths. We must then find by exact measurement how the ropes change, and from these experimentally determined laws we can develop mathematical equations which will always give the correct solution for the system. This case is exactly analogous to the Michelson-Morley experiment. We first postulate a concealed system, the stagnant ether, which always transmits vibrations with a velocity c with respect to itself. We then set up two mirrors having a velocity v with respect to the ether, so that we should measure two velocities $(c+v)$ and $(c-v)$. The experiment, however, always gives a velocity c. If we have complete faith that our concealed system (the ether) is motionless in space, we must conclude that the arms of the interferometer contract and expand according to the velocity v. This conclusion has led some physicists to reject the idea of an ether. But it can be seen from this analysis that it is no more logical to reject the ether than it would be to say that there is no machinery in Maxwell's concealed system simply because we do not know the nature of the mechanism. The correct conclusion is that the ether will not serve as an absolute basis of reference.

The Michelson-Morley experiment shows that we have no means of distinguishing between the two sets of space-time variables (x, y, z, t) and (x', y', z', t'), which have a relative velocity v. The mathematical formulation

has been given by Larmor and Lorentz in the equations *

$$x' = \beta(x - vt),\ y' = y,\ z' = z,\ t' = \beta\left(t - \frac{vx}{c^2}\right),$$

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}. \quad . \quad (2)$$

Einstein has generalized these equations by introducing observers S and S′. Thus another equation is derived from (2) by writing $-v$ for $+v$ and interchanging the primed and unprimed letters, so that

$$x = \beta(x' + vt'),\ y = y',\ z = z',\ t = \beta\left(t' + \frac{vx'}{c^2}\right). \quad . \quad (3)$$

The fundamental theorem † regarding the combination of two velocities in relativity is

$$\mathrm{W} = \frac{v + w}{1 + \dfrac{vw}{c^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This velocity is less than it would be under Newtonian mechanics. There we would have $\mathrm{W} = (v + w)$. The physical reason for this difference is that as the light-wave travels a distance w in one system, the system moves a distance wv/c with respect to the second system. Therefore the clock which is set by a light-signal from one system to the other will lag behind wv/c^2. Hence the second in one system is $(1 + wv/c^2)$ seconds in the other. This is the physical explanation of the difference between the times in the two systems.

Another important generalization of the restricted theory of relativity is the relation between mass and energy. An equation $\mathrm{E} = mc^2$ has been developed which seems to show that mass will dissolve into energy and that energy will turn into mass. This is really a Newtonian equation in which a mass m is regarded as having a velocity c. Since light-waves travel with a velocity c, it is legitimate to suppose that all the equations are unchanged by postulating that the light-wave is motionless in space and the mass m is approaching the light-waves with velocity c. This would give an energy $mc^2/2$. Since, also, the mass m could be considered as moving in the opposite direction

* Cunningham, ' The Principle of Relativity,' p. 43.
† Einstein, ' Relativity,' p. 46.

with velocity c, it would have energy $mc^2/2$ on account of this motion. Hence it is impossible by experimental methods to distinguish between an energy zero and an energy mc^2. Such a relation as this is entirely acceptable in a mathematical equation which will make a mass m pounds change in mass according to its velocity, but it will still be the same mass m with the same number of electrons, protons, neutrons, etc. There is no mechanism in the relativity theory to produce an electron or proton out of radiant energy. There is an equation which will make an already existing electron change its apparent mass as its velocity is varied. The truth of these statements will appear from the mathematical equations.

If the acceleration in a stationary system S is $a_x = du_x/dt$, the acceleration as observed from the second system S′ will be

$$a'_x = \frac{a_x\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1-\frac{vu_x}{c^2}\right)^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When $u_x = v$, we get

$$a'_x = \frac{a_x}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This means that we get from equation (6)

$$\frac{a_x}{a'_x} = (1-v^2/c^2)^{3/2}.$$

This means that the observer S′, on account of his velocity v, must change the units of length and time of S in such a way as to make the acceleration of the mass m_0, which is a_x in the S system, appear to be less than in the S′ system by a factor $(1-v^2/c^2)^{3/2}$. Hence it appears to S′ that Newton's law in S is expressed by the equation

$$\frac{m_0}{(1-v^2/c^2)^{3/2}}\frac{d^2x}{dt^2} = \mathrm{X}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The factor $1/(1-v^2/c^2)^{3/2}$ in (7) really belongs to the second factor d^2x/dt^2. If, however, we connect it with the m_0, we may say that the acceleration in S is not as great as it should be because the mass m_0 has been

increased by a factor $1/(1-v^2/c^2)^{3/2}$. It is clear that this apparent increase in mass is due not to an increase in the number of electrons, protons, etc., but to a change in the units of length and time. Such a change can produce variations of mass in already existing matter, but it cannot change the number of these entities.

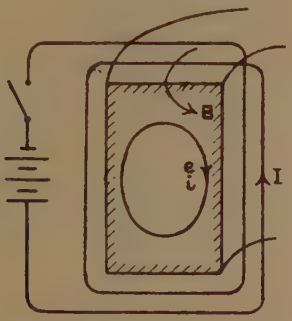
LXXIX. *On the Effect of a Magnetomotive Force applied for a short time to a Steel Cylinder.* *By* A. G. Warren, *M.Sc., M.I.E.E., F.Inst.P., and* R. G. Friend, *M.A., B.Sc., Radiological Branch, Research Department, Woolwich* *.

ONE method of inspection of steel castings and forgings involves the magnetization of the material and the spraying of the surface by a suspension of finely divided iron. Cracks which emerge to the surface are rendered clearly visible, although they may be so fine as to completely elude observation before spraying. Only the surface, of course, can be thus examined. The examination is, however, of far greater value than that fact might at first suggest. Steels, particularly alloy steels, are liable to fine cracks (or "flakes") associated with the heat treatment. These cracks do not occur singly. When conditions have been conducive to their production they are very numerous. It can therefore be assumed, with a high degree of probability, that if the surface of a steel casting or forging is, on close inspection, found to be clean, that the material is, from this point of view, sound. There may, of course, be other internal flaws, detracting from the strength of the material, but the condition mentioned is one of the most serious which can occur. The examination is superficial, and it is only necessary to magnetize the material to quite a small depth. In the examination of a large casting or forging an electromagnet may be applied to magnetize a portion, and the inspection carried out section by section. When the body to be examined is of hollow cylindrical form a convenient method of magnetizing is to loop the cylinder with a wire or coil through which a current is passed.

* Communicated by the Authors.

The cylinder is thus circularly magnetized, and there is no external evidence of magnetization except in the neighbourhood of flaws. A particular development of the method, termed "flash magnetization," has already been described by one of us *. In it advantage is taken of the fact that the retentivity of steel is high and the surface layers of a steel body forming a closed magnetic circuit may be permanently magnetized by the application of a magnetomotive force for a very short time. Potential must be applied to the coil sufficiently long for the magnetizing current to grow to the requisite value. Longer application results in the effective depth of magnetization increasing. What minimum depth is

Fig. 1.



required in practice is uncertain, but a sufficient depth has been obtained in forgings of moderate size with application of the magnetomotive force for a few milliseconds.

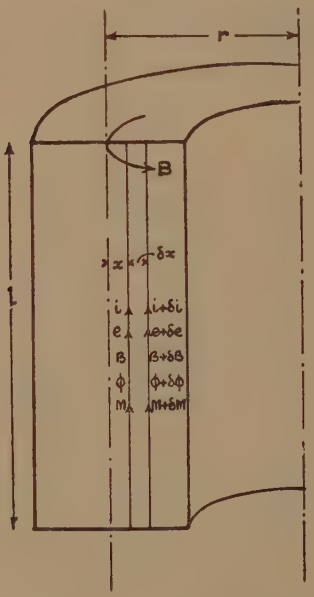
In order that magnetization can be effected with a portable coil, and that the disturbance to the electrical system to which it is connected should not be too severe, it is essential to keep the duration of the current short. The extension of the method to big forgings raised a doubt whether, in the time which could be permitted, a sufficient degree of magnetization could be obtained. It is to this question that the following analysis has given a positive answer.

The conditions are briefly these. On closing the switch (fig. 1) a current I grows in the coil, ultimately

* 'Engineering,' October 4th, 1935.

attaining the ohmic value if contact is maintained. Analysis shows that the *initial rate of growth of this current is unaffected by the presence of the iron*, but is determined only by the applied potential and the incidental inductance in circuit. The growing flux in the iron generates eddy e.m.f.'s e, which cause eddy currents i to circulate. These currents delay magnetization of the subsurface layers. In consequence the back e.m.f. generated in the mag-

Fig. 2.



netizing coil is considerably less than it would be if the iron were laminated, the rate of growth of the current I is rapid, and surface magnetization is accomplished in a very short time.

A detailed examination of the general case would be extremely laborious. Conditions are much simplified by assuming a case which is often approximately met in practice, that of a cylinder whose wall-thickness is small compared with its diameter and length.

The conditions are shown in fig. 2. The hollow cylinder of mean radius r is magnetized by a coil of T

turns carrying a current I. The length of the cylinder is l, the wall-thickness 2δ. $r>>2\delta$ and $l>>2\delta$. At a distance x from the mid-circumferential layer the magnetomotive force is M (the surface value when $x=\delta$ being M_s), the flux density B (the surface value being B_s), the flux enclosed within the area $-x$ to $+x$ is ϕ, the current density i and the e.m.f. round the area enclosed between $-x$ and $+x$ is e, this e.m.f. being the induced e.m.f. ; i is produced by e. Positive directions are anti-clockwise, as shown. Then

$$M_s=4\pi IT/10, \quad B_s=4\pi IT\mu/10.2\pi r=2IT\mu/10r,$$

$$\delta M=-(4\pi/10)2\pi ri\delta x, \quad i=e/2l\rho,$$

whence

$$\frac{\delta B}{\delta x}=-\frac{4\pi}{10}\cdot\frac{\mu e}{2l\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again

$$\delta\phi=2l\delta xB \quad \text{and} \quad e=-\frac{1}{10^8}\frac{\delta\phi}{\delta t},$$

whence

$$\frac{\delta e}{\delta x}=-\frac{1}{10^8}\frac{\delta^2\phi}{\delta t.\delta x}=-\frac{1}{10^8}\frac{\delta}{\delta t}(2lB), \quad . \quad . \quad (2)$$

giving

$$\frac{\delta e}{\delta x}=-\frac{2l}{10^8}\frac{\delta B}{\delta t} \quad \text{or} \quad \frac{\delta e}{\delta x}=PB \quad . \quad . \quad . \quad (2\,a)$$

and

$$\frac{\delta B}{\delta x}=-\frac{2\pi\mu}{10l\rho}e \quad \text{or} \quad \frac{\delta B}{\delta x}=Qe, \quad . \quad . \quad . \quad (1\,a)$$

where

$$P=-2lp/10^8, \quad Q=-2\pi\mu/10l\rho, \quad p\equiv\frac{\delta}{\delta t},$$

whence

$$\frac{\delta^2B}{\delta x^2}=PQB, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Putting

$$n^2=PQ=kp, \quad k=4\pi\mu/10^9\rho,$$

the solution of (3) is

$$B=A_1e^{nx}+A_2e^{-nx}.$$

Since B is the same for $\pm x$, $A_1=A_2$, and the solution may be written

$$B=K\cosh nx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From (2 *a*) we obtain

$$e=\mathrm{P}\int \mathrm{B}\,dx=\frac{\mathrm{P}}{n}\mathrm{K}\sinh nx \quad . \quad . \quad . \quad . \quad (5)$$

for $x=\delta$, $\mathrm{B}=\mathrm{B}_s$ or $\mathrm{B}_s=\mathrm{K}\cosh n\delta$.

Hence, under Heaviside conditions,

$$\left.\begin{aligned} \mathrm{B}&=\mathrm{B}_s\frac{\cosh nx}{\cosh n\delta}1, \\ e&=\frac{\mathrm{P}}{n}\mathrm{B}_s\frac{\sinh nx}{\cosh n\delta}1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

1 must, of course, be interpreted according to the unit function assumed.

In general the operational equation is of the form

$$w=\mathrm{X}\frac{\mathrm{Y}_p}{\mathrm{Z}_p}1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the solution of which is

$$w=\mathrm{X}\left[\frac{\mathrm{Y}_0}{\mathrm{Z}_0}+\sum_{p_1p_2\ldots}\left(\mathrm{Y}_p e^{pt}\Big/p\frac{d\mathrm{Z}_p}{dp}\right)\right]. \quad . \quad . \quad (8)$$

Case 1.

If contact is maintained for a considerable time the initial conditions are relatively unimportant, and it may be assumed that the flux at any time is approximately that which it would have been had the current suddenly assumed the constant value I at $t=0$. I, of course, defines B_s. To find the value of B :—

In (6) put $m^2=-n^2$, giving

$$\mathrm{B}=\mathrm{B}_s\frac{\cos mx}{\cos m\delta}1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The solutions of $\mathrm{Z}_p=0$ are

$$m\delta=\pm\frac{\pi}{2},\quad \pm\frac{3\pi}{2}\ldots\pm\frac{2s-1}{2}\pi\ldots$$

Since $p=-m^2/k$, and we are only concerned with the several solutions for p, we need only consider the positive values of $m\delta$.

Using $p=-m^2/k$ and $\dfrac{d\mathrm{Z}}{dp}=\dfrac{d\mathrm{Z}}{dm}\cdot\dfrac{dm}{dp}$, we obtain

$$p\frac{d\mathrm{Z}}{dp}=\frac{m\delta}{2}(-1)^s,$$

$$p_s = -\frac{(2s-1)^2}{4k} \cdot \frac{\pi^2}{\delta^2},$$

and $$Y_s = \cos\frac{2s-1}{2} \cdot \frac{\pi x}{\delta},$$

whence

$$B = B_s\left[1 + \frac{4}{\pi}\sum_{s=1,2,\ldots}(-1)^s \frac{\cos\frac{2s-1}{2}\frac{\pi x}{\delta}}{2s-1} e^{-\frac{(2s-1)^2}{4k}\cdot\frac{\pi^2}{\delta^2}t}\right]. \quad (10)$$

This is the " telegraph equation," and is well known.

Proceeding similarly to find e, and in the solution substituting $x=\delta$, we obtain the back e.m.f. generated in each turn of the magnetizing coil :

$$E = -\frac{10l\rho}{\pi\mu\delta} B_s[e^{-\alpha t} + e^{-9\alpha t} + e^{-25\alpha t} \ldots], \quad . \quad . \quad (11)$$

where $$\alpha = -\pi^2/4k\delta^2.$$

At $t=0$ we obtain $E=\infty$, an impossible condition implicit in the assumption of the current I being established instantaneously. For all but short times (11) is reasonably accurate.

Case 2.

The more important practical case of flash magnetization, when the time of contact is short, presents no difficulties in analysis, but the labour involved in evaluating a practical case is considerable. It is found that the associated inductance in the circuit plays an important part, and cannot be neglected. The conditions assumed are :—Total resistance of magnetizing circuit R, associated inductance L, potential V suddenly applied at $t=0$. The operational equation for the circuit is

$$V + TE = IR + pLI. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From (5)

$$E = \frac{P}{n} K \sinh n\delta = \frac{n}{Q} K \sinh n\delta.$$

Substituting this, n^2/k for p, and $B_s = K \cosh n\delta$, we obtain

$$V = I\left(R + L\frac{n^2}{k}\right) - \frac{Tn}{Q} B_s \tanh n\delta,$$

or $$V = I\left(R + L\frac{n^2}{k} + \frac{l\rho T^2 n}{\pi r} \tanh n\delta\right).$$

Writing $n \equiv jm$, this becomes

$$V = I\left(R - L\frac{m^2}{k} - \frac{l\rho T^2 m}{\pi r}\tan m\delta\right),$$

giving

$$I = \frac{1}{R - a\theta^2 - b\theta\tan\theta} V1, \quad . \quad . \quad . \quad . \quad (13)$$

where

$$m\delta \equiv \theta, \ a \equiv L/k\delta^2, \ b \equiv l\rho T^2/\pi r\delta.$$

Solutions have to be found for the transcendental equation

$$R - a\theta^2 - b\theta\tan\theta = 0. \quad . \quad . \quad . \quad . \quad (14)$$

Noting that $p = -a\theta^2/L$, we see that positive values of θ need only be considered. For each of the values of θ we require the values of $p\dfrac{dZ}{dp}$. A convenient form for this is

$$p\frac{dZ}{dp} = -\frac{1}{2}\left\{R + (a+b)\theta^2 + \frac{(R - a\theta^2)^2}{b}\right\} = -\phi(\theta). \quad (15)$$

The solution for the current I is then

$$I = V\left[\frac{1}{R} - \sum_{\theta_1, \theta_2 \ldots}\frac{e^{-a\theta^2 t/L}}{\phi(\theta)}\right]. \quad . \quad . \quad . \quad (16)$$

The back e.m.f. generated in the magnetizing coil is obtained by substituting (16) in (12), which gives

$$-ET = V\sum_{\theta_1, \theta_2 \ldots}\frac{(R - a\theta^2)e^{-a\theta^2 t/L}}{\phi(\theta)}. \quad . \quad . \quad (17)$$

Penetration of Flux.

To determine the flux density at any depth, use may be made of equation (13). This may be re-written

$$\beta B_s = \frac{1}{R - a\theta^2 - b\theta\tan\theta} V1, \quad . \quad . \quad . \quad (13\,a)$$

where

$$\beta = \frac{10r}{2T\mu}.$$

But

$$B = B_s\frac{\cos mx}{\cos m\delta},$$

or

$$B = \frac{\cos\frac{x}{\delta}\theta}{\beta\cos\theta(R - a\theta^2 - b\theta\tan\theta)} V1. \quad . \quad . \quad (18)$$

Such an equation usually has a solution consisting of two series corresponding to

$$(a)\ \cos\theta=0\ ;\quad (b)\ \mathrm{R}-a\theta^2-b\theta\tan\theta=0.$$

For the first group of solutions $\theta=\frac{2s-1}{2}\pi$. For each of these $\tan\theta=\infty$, and so the terms of the series are severally zero. We have, therefore, only to consider the second group giving values of θ the same as those already obtained. The several terms of the current series are multiplied by the appropriate value of $\frac{\cos\frac{x}{\delta}\theta}{\beta\cos\theta}$. From (18) $\frac{\mathrm{Y}_0}{\mathrm{Z}_0}=\frac{1}{\beta\mathrm{R}}$, and so the solution is

$$\mathrm{B}=\frac{\mathrm{V}}{\beta}\left[\frac{1}{\mathrm{R}}-\sum_{\theta_1,\theta_2\ldots}\frac{\cos\frac{x}{\delta}\theta}{\cos\theta}\cdot\frac{e^{-a\theta^2 t}}{\phi(\theta)}\right].\quad .\quad .\quad (19)$$

When t is infinite, B becomes $\frac{\mathrm{V}}{\beta\mathrm{R}}=\mathrm{B}_\infty$, as it must.

Notes upon the Solution of a Practical Case.

The theory was checked upon a gun-tube of length 700 cm., the half-thickness δ was 1·1 cm.; r was 8.75 cm. From the magnetizing curve an average value of μ of 500 was selected; ρ was assumed to be 10^{-5}. Values of R and L were determined oscillographically by switching the 250-volt line (through a fuse) on to the magnetizing loop when not passed through the tube. The values of R and L included, of course, the contributions of line and sub-station generator. The determined values were R=4 ohms and L=0·006 H. The equation for θ is

$$4-0{\cdot}007892\,\theta^2-0{\cdot}07500\,\theta\tan\theta=0.$$

The first 21 values of θ were calculated to four significant figures. This can be done in an hour or so by trial and error. Early values of θ for term n are rather less than $\pi(n-\frac{1}{2})$; for large values of n, θ is rather greater than $\pi(n-\frac{3}{2})$.

Corresponding values of p and $\phi(\theta)$ were readily obtained. For calculation of the current, tables were set out for e^{-pt} and $e^{-pt}/\phi(\theta)$ for the 21 terms for times of 1, 2, 3, 4, 5, 10, 15, and 20 milliseconds. An extension of these

tables (see equation (17)) enabled E to be determined, and a further extension (equation (19)) gave the distribution

Fig. 3.

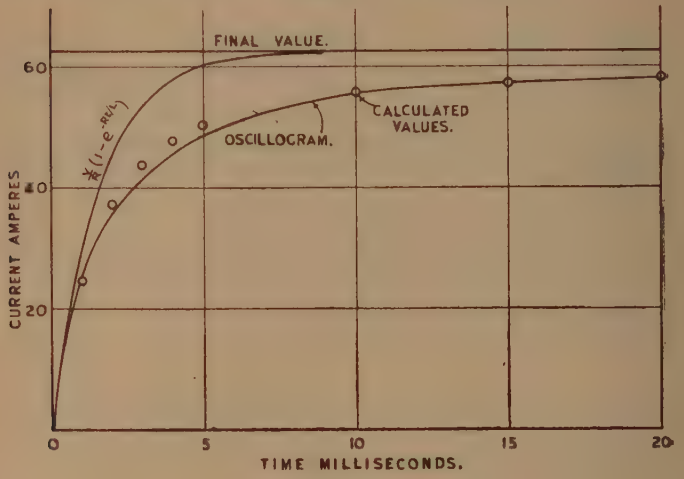
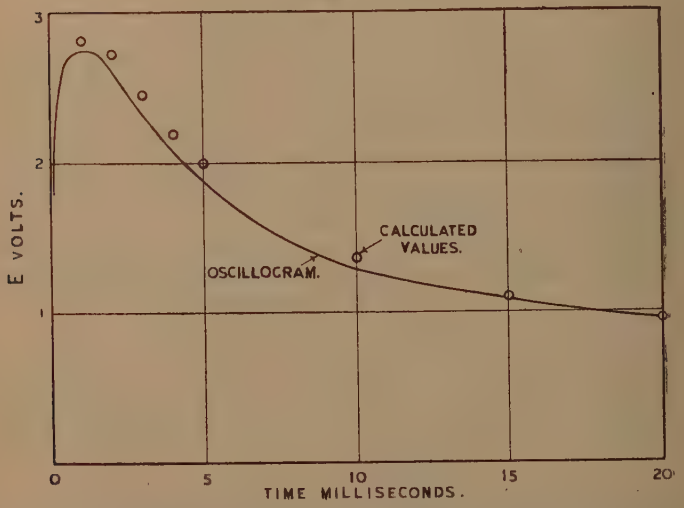


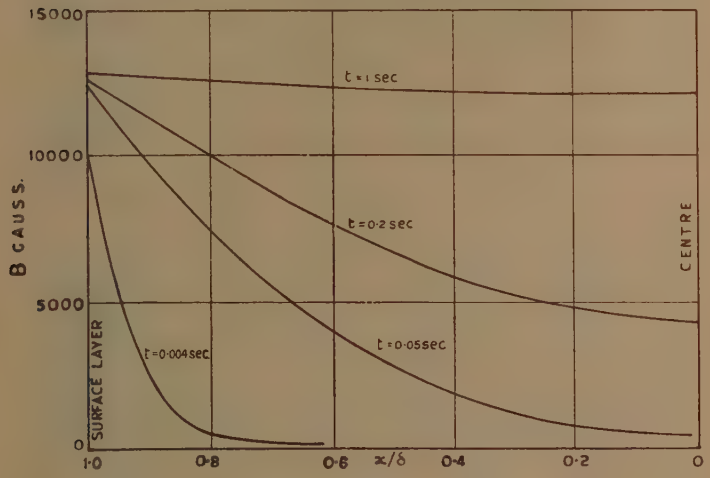
Fig. 4.



of flux at various points in the section at various times. The calculated values of I and E are shown by the circles in figs. 3 and 4. The distribution of flux is shown

in fig. 5. Figs. 3 and 4 also show the oscillograms of I and E. The agreement is as close as could be expected upon the assumption of a constant value of μ. The observed deviation may be due to the value of $\frac{dB}{dH}$ exceeding the value of $\frac{B}{H}$ over the most important part of the current range. On fig. 3 has also been plotted $\frac{V}{R}(1-e^{-Rt/L})$, giving the value of I if no iron were present. It is seen that the initial rate of rise of the current is uninfluenced

Fig. 5.



by the presence of iron, but the iron greatly delays the approach to the ohmic value. This is emphasized by fig. 4, in which it is seen that E has still a considerable value after 20 milliseconds, and again by fig. 5, which shows that even after 1 second the flux density is sensibly less at the centre of the section than it is at the surface. The calculation of the initial rate of rise of current presents some difficulty. Twenty-one terms of the series were found to be ample for the determination of the values in fig. 3. For instance, the initial value of the current is given by

$$I=V\left[\frac{1}{R}-\sum_{\theta_1,\,\theta_2\ldots}\frac{1}{\phi(\theta)}\right] \quad . \quad . \quad . \quad . \quad (20)$$

from equation (16). This should be zero, and is found to be substantially so if all terms above θ_{21} are neglected. The initial rate of rise of current is, also from equation (16),

$$\left[\frac{dI}{dt}\right]_{t=0} = \frac{V}{L} \sum_{\theta_1, \theta \ldots} \frac{a\theta^2}{\phi(\theta)} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This is equal to V/L or

$$\sum_{\theta_1, \theta_2 \ldots} \frac{a\theta^2}{\phi(\theta)} = 1. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

So far the general proof of equation (22) has eluded us, although certain special cases offer no difficulty. The terms in equation (22) converge much less rapidly than those in equation (20). In consequence it is necessary to go beyond θ_{21} to obtain an accurate estimate of $\left[\frac{dI}{dt}\right]_{t=0}$. In the case of the gun-tube considered,

$$\sum_{\theta_1 \text{ to } \theta_{21}} \frac{a\theta^2}{\phi(\theta)} = 0{\cdot}909 \quad \text{and} \quad \sum_{\theta_1 \text{ to } \theta_{41}} \frac{a\theta^2}{(\phi)\theta} = 0{\cdot}964.$$

Successive terms in the series decrease rather more rapidly than the inverse square, and rather less rapidly than exponentially. The remainder may therefore be "bracketed." So doing, we find estimating the remainder after 21 terms gives the value of the sum as between 0·955 and 1·035, and estimating the remainder after 41 terms gives the value as between 0·982 and 1·015. These calculations appear sufficient to assume that the true value is unity. In other words, the presence of the iron does not affect the initial rate of rise of the current.

The intensity of the eddy currents in the steel is of some interest. The maximum value of E is approximately 2·8 volts, which gives for the maximum density of the eddy currents at the surface 200 amperes per sq. cm.

Acknowledgments.

The authors wish to thank the Director of Radiological Research for his interest in this work. The paper is published by permission of the Director of Artillery.

LXXX. *On Lamé's Equation.* *By* J. L. Sharma, *M.A.**

1. In connexion with the problem of determining the steady temperature in an ellipsoidal conductor with prescribed temperature on its boundaries very frequently we come across Lamé's differential equation, viz.,

$$(\mu^2-h^2)(\mu^2-k^2)\frac{d^2y}{d\mu^2}+\mu(2\mu^2-h^2-k^2)\frac{dy}{d\mu}$$
$$+[(h^2+k^2)p-n(n+1)\mu^2]y=0.$$

Starting from its Weierstrassian form, namely

$$\frac{d^2y}{du^2}=[n(n+1)\wp(u)+\mathrm{B}]y, \quad . \quad . \quad . \quad (1.1)$$

Halphen† has shown that this equation admitsa solution of the form

$$\mathrm{F}_n(u)=\prod_{r=1}^{n}\left[\frac{\sigma(a_r\pm u)}{\sigma(\pm u)\sigma(a_r)}\exp\{\pm u\zeta(a_r)\}\right], \quad . \quad (1.2)$$

where n is an integer and B is arbitrary. The object of the present paper is to show that these functions, like many others occurring in harmonic analysis, possess an orthogonal property which enables us to expand an arbitrary function of u in terms of these functions. To distinguish the solutions (1.2) from Lamé's functions E_n^m we shall call them generalized Lamé's functions of order n, and represent them as ϕ_n, where

$$\phi_n=\frac{\mathrm{F}_n(u)}{\sqrt{2h_{2n+1}}}, \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

h_{2n+1} being the coefficient of u^{n+1} in $\mathrm{F}_n(u)$, so that the coefficient of u in ϕ_n^2 is unity.

2. Let ϕ_m and ϕ_n be two generalized Lamé's functions of order m and n; we have then

$$\phi_m''=[m(m+1)\wp(u)+\mathrm{B}]\phi_m,$$
$$\phi_n''=[n(n+1)\wp(u)+\mathrm{B}]\phi_n;$$

* Communicated by the Author.

† Halphen, 'Fonctions Elliptiques,' ii. (Paris, 1888), and *vide* Hermite, *Comptes Rendus*, lxxxv. (1877).

multiply the first equation by ϕ_n and the second by ϕ_m, and subtract ; we get on integration

$$(m-n)(m+n+1)\int \wp(u)\phi_m\phi_n\,du=\int^{\frac{d}{du}}(\phi'_m\phi_n-\phi_m\phi'_n)\,du. \quad . \quad . \quad . \quad (2.1)$$

Now ϕ_m, ϕ'_m, ϕ_n, ϕ'_n are all uniform functions of u and have only one pole in the period parallelogram, viz., the origin. Hence the integral of the right-hand side in (2.1) round any closed contour not passing through the pole will be zero. Therefore

$$\int_c \wp(u)\phi_m\phi_n\,du=0, \quad m\neq n, \quad . \quad . \quad . \quad (2.2)$$

taken round the same contour. Since e_1, e_2, e_3 are all real, one of the periods, namely $2\omega_1$, is purely real, and the other, namely $2\omega_3$, is purely imaginary. For convenience of computation we choose the contour as a rectangle formed by joining the points $(\pm\omega_1\pm\omega_3)$.

The contour is such that ϕ_n has only one pole within it.

3. Consider now the two fundamental solutions in series of the differential equation (1.1),

$$y_1=\frac{1}{u^n}+\frac{h_2}{u^{n-2}}+\frac{h_4}{u^{n-4}}+\ldots\ldots,$$

$$y_2=u^{n+1}+h'_2u^{n+3}+h'_4u^{n+5}+\ldots\ldots$$

So ϕ_n, being a solution, will be of the form

$$c_1y_1+c_2y_2,$$

where c_1 and c_2 are arbitrary constants. Hence up to u^{n+1} only even powers will occur if n is even and odd powers only if n is odd. Hence the residue of $\wp(u)\phi_n^2$ within the said contour is unity according to the definition (1.3). Hence by the Cauchy's theorem

$$\int_c \wp(u)\phi_n^2\,du=2\pi i. \quad . \quad . \quad . \quad . \quad (3.1)$$

4. To expand u^n.

Let $u^n=a_0\phi_0+a_1\phi_1+a_2\phi_2+\ldots\ldots+a_n\phi_n+\ldots\ldots$

Multiply both the sides by $\wp(u)\phi_p$ and integrate; we have

$$a_p=\frac{1}{2\pi i}\int u^n\wp(u)\phi_p\,du,$$

$$=0, \quad p\leqslant n-2,$$

$$=j_p^n, \quad n-p \text{ is odd},$$
$$=0, \quad n-p \text{ is even},$$

where

$$j_p^n = \sum_{k=0}^{k=\frac{p+1-n}{2}} c_{2k} h_{p+1-n-2k}$$
$$= \frac{1}{\lfloor p+1-n} \left[\frac{d^{p+1-n}}{du^{p+1-n}} (u^{p+2} \wp(u) \phi_p) \right]_{u=0}.$$

As a particular case we get

$$1 = \sum_0^\infty j_{2r+1} \phi_{2r+1},$$

where

$$j_{2r+1} = \frac{1}{\lfloor 2r+2} \left[\frac{d^{2r+2}}{du^{2r+2}} (u^{2r+3} \wp(u) \phi_{2r+1}) \right]_{u=0}.$$

5. To expand $[\wp(u)]^n$.

Let $$[\wp(u)]^n = \Sigma a_r \phi_r.$$

Multiply both sides by $\wp(u)\phi_r$ and integrate; we get

$$a_r = \frac{1}{2\pi i} \int_c [\wp(u)]^{n+1} \phi_r \, du$$
$$=0 \text{ (if } r \text{ is even and } > 2n),$$
$$\text{or } \frac{1}{\lfloor 2n+r+1} \left[\frac{d^{2n+r+1}}{du^{2n+r+1}} (u^{2n+2+r} [\wp(u)]^{n+1} \phi_r) \right]_{u=0}.$$

6. If it be assumed that the series

$$a_0\phi_0 + a_1\phi_1 + a_2\phi_2 + \ldots + a_r\phi_r + \ldots$$

converges uniformly in the fundamental period parallelogram (taken as the contour) to a function $f(u)$, we can multiply the terms of the series by $\wp(u)\phi_n$ and integrate term by term within the said contour, and equate the result to

$$\int_c f(u) \wp(u) \phi_n \, du;$$

using the fundamental theorems (2.2) and (3.1) we have

$$\int_c f(u) \wp(u) \phi_n \, du = a_n \int_c \wp(u) \phi_n^2 \, du$$
$$= 2\pi i a_n;$$

$$\therefore \quad a_n = \frac{1}{2\pi i}\int_c f(u)\wp(u)\phi_n\,du$$

$$= \frac{1}{\lfloor n+r+1}\left[\frac{d^{n+r+1}}{du^{n+r+1}}(u^{n+r+2}\wp(u)f(u)\phi_n)\right]_{u=0}, \qquad . \quad . \quad . \quad (6.1)$$

where the principal part near the origin of $f(u)$ is of degree $-r$ and it has only one pole, if at all, at the origin. Thus the series

$$\Sigma\phi_n \frac{1}{\lfloor n+r+1}\left[\frac{d^{n+r+1}}{du^{n+r+1}}(u^{n+r+2}\wp(u)\phi_n f(u)\right]_{u=0}$$

converges uniformly in the fundamental cell to the function $f(u)$. No assumption can be made in general as regards the convergence of the series which has above been assumed to converge uniformly to $f(u)$. This, however, involves several conditions to be satisfied by $f(u)$. Among these the continuity of $f(u)$, as can easily be seen, is neither necessary nor sufficient. The coefficients in this series have definite values, if $f(u)$ is completely defined, provided it be analytic within the period parallelogram except at a finite number of points where it has poles of definite order. As in the case of the ordinary Fourier and Legendre series, the above series can be shown to have definite properties. Sufficient conditions may be obtained for the convergence of the series at the prescribed point of the parallelogram or at all the points of a subcell.

7. If the function has singularities other than the origin then the value of the coefficient a_n in (6.1) will be slightly changed and will be given by

$$a_n = \frac{1}{\lfloor n+r+1}\left[\frac{d^{n+r+}}{du^{n+r+1}}(u^{n+r+2}\wp(u)f(u)\phi_n)\right]_{u=0}$$

$$+ \sum_a \frac{df(a)}{da}\cdot\wp(a)\phi_n(a),$$

where Σ extends to all the singularities.

LXXXI. *A Simple Solution of the Flat Plate Problem of Skin Friction and Heat Transfer.* *By* N. A. V. Piercy, *D.Sc.*, *M.Inst.C.E.*, *and* J. H. Preston, *B.Sc.*, *Queen Mary College* *.

1. *Introduction.*

UNDER steady conditions, and while variation of physical constants may be neglected, theoretical knowledge of skin friction and heat transfer is now complete for the flat plate at large Reynolds numbers. It will be recalled that Boussinesq [1] as early as 1901 calculated the heat transfer to an inviscid stream, although the correctness of his result only became apparent when one of the present authors and Winny [2] obtained the appropriate solution of King's [3] integral equation for the heat sources in 1933. This verification has just been corroborated in the course of a mathematically exact solution obtained by Bickley [4], using a third method. Turning to viscous flow, Blasius [5] in 1908 successfully computed the skin friction from Prandtl's boundary layer equations of motion, his result being checked by Topfer [6] in 1912. Fage and Falkner [7] calculated the corresponding heat transfer in 1931 by the same method. Again, starting from Bairstow and Misses Cave and Lang's integral equation [8], one of the present authors, with Winny [9] found in 1933 the asymptotic skin friction according to Oseen's approximation to the equations of motion and, finally, in collaboration with Schmidt [10] in 1934, the corresponding solution for the heat transfer. A theorem in the last paper enabled paper [7] to be checked.

Apart from analytical interest, the case has warranted this detailed attention notably for two reasons. First may be mentioned its reality, turbulence not occurring with small temperature differences until the motion reaches a large scale. Solutions given in papers [5], [7], and [10] give fair agreement with experiment shortly before unsteadiness intervenes, discrepancies ranging from 6 to 20 per cent. Secondly, the flat plate forms a convenient starting-point for developments of the subject along approximate lines leading to applications of technical importance.

* Communicated by the Authors.

It is here that solutions in papers [9] and [10] are especially useful on account of their analytical convenience. Little of the work is superseded. But so divergent have been the various angles from which the subject has been approached that the present state of the theory must be described as heterogeneous. Moreover, the bulk of literature on which it rests is not of the easiest to read on account of analytical or computative intricacy.

The present paper aims first at a homogeneous treatment, yielding all existing approved results, while avoiding the complications of the original investigations. It will be understood from the first sentence that only asymptotic conditions are considered, and it is also deemed sufficient to limit the thermal boundary condition to that of uniform temperature on the plate.

We begin *à priori*, writing down the appropriate form of Prandtl's equations and those for the temperature at a point, and making use of a transformation of co-ordinates suggested by Blasius. Development then follows by a method of successive approximation, which is by no means new, though apparently not widely known in Hydrodynamics. Brief search reveals its employment by Helmholtz [11] in a vibration problem, a use whose legitimacy remained for some time in doubt until finally established by Poincaré and others [12]. It is again found suggested [13] as a means of improving Stokes's solution for the sphere, where it failed, however. Thus success in application to a given problem cannot always be assured beforehand.

Effective contribution to the general theory of the method is beyond the scope of the present paper. But a second aim of the paper is to call attention again to the method, by way of the present well-documented case, in the hope that it may prove useful in other investigations. This hope, it may be added, is beginning to appear well founded.

2. *General Method.*

It is assumed with regard to the fluid that the density ρ, the kinematic coefficient of viscosity ν, the thermal conductivity k, and the heat capacity per unit volume c remain constant and that at any point the velocity components u and v and the temperature θ take up steady values.

The undisturbed velocity of the stream is assumed constant and is denoted by U and the length of the plate by l, so that the Reynolds number $R = Ul/\nu$. The boundary conditions at the plate are taken as $u = v = 0$ and $\theta = \theta_0$, a constant, and at ∞ $u = U$, $v = \theta = 0$.

The assumption of large Reynolds numbers means that a thin boundary layer exists and, further, that the pressure may be regarded as constant throughout the fluid. Hence Prandtl's equations for the flow reduce to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From similar considerations the equations for the temperature at any point become

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{k}{c}\frac{\partial^2 \theta}{\partial y^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If u_1, v_1 are known as first approximations to u and v, the equation

$$u_1\frac{\partial u_2}{\partial x} + v_1\frac{\partial u_2}{\partial y} = \nu\frac{\partial^2 u_2}{\partial y^2} \quad . \quad . \quad . \quad . \quad (3)$$

may be regarded as an equation for determining a second approximation u_2 for this velocity component, whence v_2, a second approximation to v, will follow from the equation of continuity, the boundary conditions being duly satisfied.

On comparison (2) and (3) are seen to be the same, but they are to satisfy different boundary conditions. Moreover, the degree of approximation is different. It is not particularly convenient to consider the transference of heat and momentum together, and the skin friction will be evaluated first.

3. *The Skin Friction.*

We have from (3) for the nth approximation to u and v

$$u_{n-1}\frac{\partial u_n}{\partial x} + v_{n-1}\frac{\partial u_n}{\partial y} = \nu\frac{\partial^2 u_n}{\partial y^2}, \quad . \quad . \quad . \quad (4)$$

together with

$$\frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial y} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Taking the axis of x in the plane of the plate with the origin at its nose, we have for the boundary conditions: $u_n = v_n = 0$ for $y = 0$ between $x = 0$ and l; while at infinity $u_n = \mathrm{U}$ and $v_n = 0$.

Earlier work suggests the substitution $u_n = u_n/\mathrm{U}$ and transformation to the coordinates

$$\xi = \left(\frac{\mathrm{R}y^2}{4xl}\right)^{\frac{1}{2}}, \qquad \eta = \frac{x}{l}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The boundary conditions then become $u_n = 0$ on the plate and $= 1$ at ∞, while, on assuming that $\partial u_n/\partial\eta = 0$ gives (as this is known and can be verified to do) a solution satisfying the boundary conditions, (4) reduces to

$$\left(-u_{n-1}\frac{\xi}{2} + v_{n-1}\sqrt{\frac{\mathrm{R}\eta}{4}}\right)\frac{d\bar{u}_n}{d\xi} = \frac{1}{4}\frac{d^2\bar{u}_n}{d\xi^2}. \quad . \quad . \quad (7)$$

In terms of the new coordinates and with the assumption mentioned the equation of continuity transforms to

$$-\xi\frac{d\bar{u}_n}{d\xi} + \sqrt{\mathrm{R}\eta}\,.\frac{dv_n}{d\xi} = 0, \quad . \quad . \quad . \quad . \quad (8)$$

whence

$$v_{n-1} = (\mathrm{R}\eta)^{-\frac{1}{2}}\int_0^\xi \xi\frac{d\bar{u}_{n-1}}{d\xi}\,d\xi + \mathrm{B}, \quad . \quad . \quad (9)$$

and integration by parts gives from the boundary condition at the plate $\mathrm{B} = 0$ and

$$v_{n-1} = (\mathrm{R}\eta)^{-\frac{1}{2}}\left(\xi u_{n-1} - \int_0^\xi u_{n-1}\,d\xi\right). \quad . \quad . \quad (10)$$

Writing

$$f_{n-1}(\xi) \equiv -2\int_0^\xi u_{n-1}\,d\xi \quad . \quad . \quad . \quad . \quad (11)$$

and substituting from (10) reduces (7) further to

$$\frac{d^2u_n}{d\xi^2} = f_{n-1}(\xi)\frac{du_n}{d\xi}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Integrating twice,

$$\bar{u}_n = \mathrm{A}_n\int_0^\xi e^{\int_0^\xi f_{n-1}(\xi)\,d\xi}\,d\xi + \mathrm{C}, \quad . \quad . \quad . \quad (13)$$

where, from the boundary conditions, $\mathrm{C} = 0$ and

$$\frac{1}{\mathrm{A}_n} = \int_0^\infty e^{\int_0^\xi f_{n-1}(\xi)\,d\xi}\,d\xi. \quad . \quad . \quad . \quad . \quad (14)$$

Regarding the irrotational flow as viscous flow with an infinitely thin boundary layer, when the plate becomes a double surface of bound vortex lines, the skin friction F has infinity as a first approximation and is given to the nth approximation by

$$\begin{aligned} \mathrm{F}_n &= 2\mu \int_0^l \left(\frac{\partial u_n}{\partial y}\right)_{y=0} dx \\ &= 2\mu \mathrm{U} \int_0^l \left(\frac{\mathrm{R}}{4xl}\right)^{\frac{1}{2}} \left(\frac{\partial \bar{u}_n}{\partial \xi}\right)_{\xi=0} dx. \quad . \quad . \quad . \quad (15) \end{aligned}$$

Introducing the drag coefficient k_{D},

$$k_{\mathrm{D}n} \equiv \frac{\mathrm{F}_n}{\rho \mathrm{U}^2 l} = \frac{2\mathrm{A}_n}{\sqrt{\mathrm{R}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

from (13) and (14).

We could proceed directly to the computation of A_n, but there is an interest in evaluating the second and third approximations separately, as we proceed to do.

4. *Asymptotic Solution to Oseen's Approximation.*

The first approximation is that the velocities are known only for inviscid flow, *i. e.*, $u_1 = \mathrm{U}$, $v_1 = 0$. From (3) we then have

$$\mathrm{U} \frac{\partial u_2}{\partial x} = \nu \frac{\partial^2 u_2}{\partial y^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which is Oseen's approximation to the equations of motion past a flat plate at large Reynolds numbers, when the pressure is constant and the second term of $\nabla^2 u_2$ may be neglected in comparison with the first.

Since u_1 now equals unity, (14) gives at once on taking account of (11)

$$\mathrm{A}_2 = 2/\sqrt{\pi}.$$

Hence

$$\bar{u}_2 = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-z^2} dz = \mathrm{erf}\,(\xi). \quad . \quad . \quad . \quad (18)$$

for which tables exist, and

$$k_{\mathrm{D}2} = 2\mathrm{A}_2/\mathrm{R}^{\frac{1}{2}} = 4/(\pi \mathrm{R})^{\frac{1}{2}} = 2{\cdot}257\ \mathrm{R}^{-\frac{1}{2}}. \quad . \quad . \quad (19)$$

These results agree with those obtained by one of the authors with Winny [9], by a much more complicated

method. The degree of physical approximation is fully discussed in the paper mentioned; according to a more recent experimental analysis by Fage [13] the result (19) is excessive by about 60 per cent.

5. *Third Approximation.*

The velocity components u_2, v_2 determined in § 4 are now inserted in (3) to form an equation for u_3.

From (18), (14), and (11) we have

$$\frac{1}{A_3} = \int_0^\infty e^{-\frac{4}{\sqrt{\pi}}\int_0^\xi \int_0^\xi \int_0^\xi e^{-z^2}\,dz\,d\xi\,d\xi}\,d\xi.$$

Integrating by parts reduces this to

$$\frac{1}{A_3} = \int_0^\infty e^{\frac{2}{\sqrt{\pi}}\int_0^\xi \left(1-e^{-\xi^2}-2\xi\int_0^\xi e^{-z^2}dz\right)d\xi}\,d\xi,$$

an expression which may be checked from paper [10] and further to

$$\frac{1}{A_3} = \int_0^\infty e^{\frac{2}{\sqrt{\pi}}(1-\frac{1}{2}e^{-\xi^2})\xi-(\frac{1}{2}+\xi^2)\operatorname{erf}(\xi)}\,d\xi, \quad . \quad . \quad (20)$$

a form which is convenient for numerical calculation if high accuracy is desired.

Retaining only the first term of the expansion of e^{-z^2} gives at once

$$\frac{1}{A_3} = \int_0^\infty e^{-\frac{4}{\sqrt{\pi}}\cdot\frac{\xi^3}{6}}\,d\xi = \left(\frac{3\sqrt{\pi}}{2}\right)^{\frac{1}{3}}\Gamma(\tfrac{4}{3}), \quad . \quad . \quad (21)$$

so that

$$k_{D3} = \frac{2}{\Gamma(\frac{4}{3})}\left(\frac{2}{3\sqrt{\pi}}\right)^{\frac{1}{3}}.\,R^{-\frac{1}{3}} = 1{\cdot}615\,R^{-\frac{1}{3}}. \quad . \quad . \quad (22)$$

The coefficient of $R^{-\frac{1}{3}}$ in (19) is exact. But that in (22) is in error to an extent that is readily determined by numerical evaluation of (20). In this way we find

$$k_{D3} = 1{\cdot}512\,R^{-\frac{1}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22\,a)$$

to compare with (22), so that the error is 7 per cent.

It is not destructive to the final approximation to retain this error.

6. *Ultimate Approximation to the Skin Friction.*

It is now convenient to write

$$\left.\begin{aligned} m_2 &= 4/\sqrt{\pi}, \\ m_3 &= \frac{2}{\Gamma(\frac{4}{3})}\left(\frac{m_2}{6}\right)^{\frac{1}{3}}, \\ m_4 &= \frac{2}{\Gamma(\frac{4}{3})}\left(\frac{m_3}{6}\right)^{\frac{1}{3}}, \\ &\cdots\cdots\cdots \\ m_n &= \frac{2}{\Gamma(\frac{4}{3})}\left(\frac{m_{n-1}}{6}\right)^{\frac{1}{3}}. \end{aligned}\right\} \quad \ldots\ldots (23)$$

The 4th and subsequent approximations on taking only first terms reduce to the same form, so that we have from the foregoing

$$\left.\begin{aligned} k_{D2} &= m_2 R^{-\frac{1}{2}}, \\ k_{D3} &= m_3 R^{-\frac{1}{2}}, \\ &\cdots\cdots \\ k_{Dn} &= m_n R^{-\frac{1}{2}}. \end{aligned}\right\} \quad \ldots\ldots (24)$$

Now

$$\left.\begin{aligned} m_4 &= \frac{2}{\Gamma(\frac{4}{3})}\left\{\frac{1}{3\Gamma(\frac{4}{3})}\left(\frac{m_2}{6^{\frac{1}{3}}}\right)^{\frac{1}{3}}\right\}^{\frac{1}{3}} \\ &= \frac{2}{\Gamma(\frac{4}{3})}\cdot\frac{1}{[3\Gamma(\frac{4}{3})]^{\frac{1}{3}}}\left(\frac{m_2}{6}\right)^{(\frac{1}{3})^2} \\ &\cdots\cdots\cdots \\ m_n &= \frac{2}{\Gamma(\frac{4}{3})}\left(\frac{m_2}{6}\right)^{(\frac{1}{3})^{n-2}}/[3\Gamma(\frac{4}{3})]^{\frac{1}{3}+(\frac{1}{3})^2+\ldots(\frac{1}{3})^{n-3}} \end{aligned}\right\}, \quad \ldots (25)$$

whence

$$m_\infty = \frac{2}{\Gamma(\frac{4}{3})}\cdot\frac{1}{[3\Gamma(\frac{4}{3})]^{\frac{1}{3}}}. \quad \ldots\ldots (26)$$

Hence for the ultimate approximation to the drag coefficient obtainable by the present method we have

$$k_D = m_\infty R^{-\frac{1}{2}} = 1{\cdot}367\ R^{-\frac{1}{2}}. \quad \ldots\ldots (27)$$

The elaborate solution of (1) by Blasius as corrected by Töpfer gives

$$k_D = 1{\cdot}328\ R^{-\frac{1}{2}}. \quad \ldots\ldots (28)$$

The most recent assessment of the mean experimental coefficient of $R^{-\frac{1}{2}}$ under steady conditions at high Reynolds numbers [14] is 1·40. This is subject to an error, according to the evidence on which it is founded, of about 4 per cent. The simply derived result (27) is within 3 per cent. of either mean experiment or accurate computation.

It will be possible to improve the accuracy of (27) with little trouble. Before considering this, however, it is convenient to develop the corresponding analysis for the heat transfer.

7. *First Approximation to Heat Transfer.*

The present method applied to heat transfer gives as a first step that to inviscid flow. Putting $u=U$, $v=0$, and introducing θ defined by

$$\bar{\theta}=(\theta_0-\theta)/\theta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

changes (2) to

$$U\frac{\partial\bar{\theta}_1}{\partial x}=\frac{k}{c}\frac{\partial^2\theta_1}{\partial y^2}. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Introducing the coordinates

$$\xi=\left(\frac{cU}{k}\cdot\frac{y^2}{4x}\right)^{\frac{1}{2}}, \quad \eta=\frac{x}{l}, \quad . \quad . \quad . \quad (31)$$

and assuming that $\partial\theta/\partial\eta=0$ gives an appropriate solution, we obtain

$$\frac{d^2\bar{\theta}_1}{d\xi^2}=-2\xi\frac{d\bar{\theta}_1}{d\xi} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

of which the integral is

$$\bar{\theta}_1=A_1'\int_0^{\xi}e^{-\xi^2}d\xi+C. \quad . \quad . \quad . \quad . \quad (33)$$

The thermal boundary conditions are $\theta_1=0$ on the plate and unity at ∞. Hence $C=0$ and

$$\frac{1}{A_1'}=\int_0^{\infty}e^{-\xi_1^2}d\xi_1, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

so that

$$A_1'=2/\sqrt{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The first approximation H_1 to the heat loss from both sides of the plate is

$$H_1 = -2k\int_0^l \left(\frac{\partial\theta}{\partial y}\right)_{y=0} dx$$

$$= 2k\theta_0\int_0^l \left(\frac{cU}{k}\cdot\frac{1}{4x}\right)^{\frac{1}{2}}\left(\frac{\partial\theta}{\partial\xi}\right)_{\xi=0} dx, \quad . \quad (36)$$

giving from (33)

$$\frac{H_1}{k\theta_0} = \frac{4}{\sqrt{\pi}}\left(\frac{cUl}{k}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (37)$$

This result is identical with that obtained first by Boussinesq, and subsequently by elaborate methods in papers [2] and [4]. It will be noted that the coefficient on the right-hand side is m_2.

8. *Second Approximation.*

This is found, in principle, by substituting from (18) and the equation of continuity in (2). Since these approximations to the velocity components accord with those assumed in paper [10], and there is no change in the method of solution, the second approximation to H is exactly as there given. It may be written in the form

$$\frac{H_2}{k\theta_0} = \frac{2}{\Gamma(\frac{4}{3})}\left(\frac{2}{3\sqrt{\pi}}\right)^{\frac{1}{2}}\left(\frac{c\nu}{k}\right)^{\frac{1}{2}} R^{\frac{1}{2}} = m_3\left(\frac{c\nu}{k}\right)^{\frac{1}{2}} R^{\frac{1}{2}}. \quad . \quad (38)$$

9. *Ultimate Approximation to H.*

Generalizing this method of attack it is easily verified that

$$\frac{H_n}{k\theta_0} = m_{n+1}\left(\frac{c\nu}{k}\right)^{\frac{1}{2}} R^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (38)$$

so that ultimately, as for (26),

$$\frac{H}{k\theta_0} = \frac{2}{\Gamma(\frac{4}{3})}\frac{1}{[3\Gamma(\frac{4}{3})]^{\frac{1}{2}}}\left(\frac{c\nu}{k}\right)^{\frac{1}{2}} R^{\frac{1}{2}}. \quad . \quad . \quad . \quad (40)$$

Adopting the mean experimental value 1·3 for $c\nu/k$, we have for gases

$$H/k\theta_0 = 1{\cdot}252\ R^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

This result is within 1 per cent. of that obtained in paper [7] by more complicated methods (1·240).

Actually, it is believed that (41) is excessive by 3 per cent., *i. e.*, that the same error occurs as in (27). At

first sight this may appear discordant with a theorem proved in paper [10] with Schmidt, by which for a given approximation in u and v the heat transfer is in considerably less error than the skin friction. But that the present analysis agrees with the theorem is readily seen as follows :

$$\frac{H_n}{H_{n-1}} = \frac{m_{n+1}}{m_n} = \left(\frac{m_n}{m_{n-1}}\right)^{\frac{1}{3}} = \left(\frac{k_{\theta n}}{k_{\theta n-1}}\right)^{\frac{1}{3}} \quad . \quad . \quad . \quad (42)$$

10. It will be obvious that the above theorem might be applied to obtain successive approximations to the skin friction. But we note that two successive solutions would be required initially and also that the accuracy achieved would not be improved.

The accuracy of (27) and (41) appears remarkable when it is considered that each successive approximation is derived from its predecessor by a process which is itself approximate. The second cause of error can be eliminated, of course, by numerical integration, for which a method is provided in § 5. Following this we have prepared the table below to show the convergence of the accurately evaluated approximations. Those integrated from first terms only are added for comparison. The friction coefficient has been chosen for illustration.

Values of $k_D R^{\frac{1}{3}}$.

No. of approximation	2	3	4	5	6
Taking first terms only	—	1·615	1·445	1·392	1·375
By numerical integration ..	2·257 —	1·512	1·374	1·341	1·331

The sixth approximation obtained numerically is within $\frac{1}{4}$ per cent. of the accurate value (1·328), which it appears would probably be reproduced by the 8th approximation.

11. *Conclusion.*

A new solution is obtained of the flat plate problem of skin friction and heat transfer under steady conditions at large Reynolds numbers. The method employed is one of successive approximation, and the several results of the past 35 years, on which the theory of the subject rests, are reproduced as steps. In contrast with earlier investigations, the analysis is throughout simple and the

need of computation slight. With no numerical work at all it is possible to attain within 3 per cent. of the final results of either more exact analysis or mean experiment. It is hoped that the method will find other uses in Hydrodynamics. Its theory is not discussed from a general point of view, but convergence is shown to be so rapid in the present application that the sixth approximation approaches the correct analytical result within 0·25 per cent.

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LXXXII. *Note on the Bending of Rectangular Plates.* *By* J. C. COSTELLO, *M.E.**

IN the Phil. Mag. vol. xx. no. 134, pp. 607–610, E. H. Bateman has considered the problem of flexure of a rectangular plate, loaded uniformly, and supported by a large number of rigid pillars marking out rectangular panels, the pillars being joined by beams which afford a flexible support to the edges. A panel not near the edge is represented in fig. 1; at the four corners A, B, C, D, supported by pillars, the transverse deflexion w is zero; along the edges AB, CD, $w \neq 0$ and $\frac{\partial w}{\partial y} = 0$, along BC, DA, $w \neq 0$ and $\frac{\partial w}{\partial x} = 0$.

* Communicated by the Author.

Bateman gives as the solution:

$$w = w_0 + w_1 + w_3 + w_5 \ldots,$$

where $$w_0 = \frac{P}{48D}\{x^2-a^2)^2+(y^2-b^2)^2\},$$

and satisfies the elastic equation $\nabla^4 w = P/D$ and the boundary conditions, while the complementary functions w_n which satisfy the equation $\nabla^4 w = 0$ and the boundary conditions, are defined by

$$w_n = A_n\left\{\left(L_n \cosh\frac{n\pi x}{b} + \frac{x}{a}\sinh\frac{n\pi x}{b}\right)\cos\frac{n\pi y}{b} + L_n\cosh\frac{n\pi a}{b} + \sinh\frac{n\pi a}{b}\right\}$$

$$+B_n\left\{\left(M_n\cosh\frac{n\pi y}{a} + \frac{y}{b}\sinh\frac{n\pi y}{a}\right)\cos\frac{n\pi x}{a} + M_n\cosh\frac{n\pi b}{a} + \sinh\frac{n\pi b}{a}\right\},$$

where $$L_n = -\left(\frac{b}{n\pi a} + \coth\frac{n\pi a}{b}\right),$$

and $$M_n = -\left(\frac{a}{n\pi b} + \coth\frac{n\pi b}{a}\right).$$

The constants A_n B_n are determined by the edge deflexion and can be adjusted to give any arbitrary deflexion of the supporting beams. In particular, where the supporting beams are absent, or of zero stiffness, the constants A_n, B_n are zero.

The solution in the form given is not, however, correct (except to a first approximation), because the absence of the cosines of even multiples of $\frac{\pi x}{a}$ and $\frac{\pi y}{b}$ renders it impossible for the function w to represent exactly the edge deflexion in the interval $-a<x<a$ $(y=b)$ or $-b<y<b$ $(x=a)$. This is apparent on considering the special case of rigidly supported edges.

The corrected form of the solution is

$$w = w_0 + \sum_{n=1}^{\infty} A_n\left\{\left(L_n\cosh\frac{n\pi x}{b} + \frac{x}{a}\sinh\frac{n\pi x}{b}\right)\cos\frac{n\pi y}{b} - \cos n\pi\left(L_n\cosh\frac{n\pi a}{b} + \sinh\frac{n\pi a}{b}\right)\right\}$$

$$B\left\{\left(M_n \cosh\frac{n\pi y}{a}+\frac{y}{b}\sinh\frac{n\pi y}{b}\right)\cos\frac{n\pi x}{a}\right.$$
$$\left.-\cos n\pi\left(M_n\cosh\frac{n\pi b}{a}+\sinh\frac{n\pi b}{a}\right)\right\},$$
. . . (1)

where w_0, L_n, M_n are as given above.

The constants A_n B_n are best determined by harmonic analysis, as numerical trial indicates that the process of matching the deflexions is not quite satisfactory.

For the square plates the harmonic analysis required is

$$w_0=\frac{pa^4}{48D}\left[\frac{8}{15}-\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^4}\cos\frac{n\pi x}{a}\right]$$
$$=\alpha_0+\sum_1^{\infty}\alpha_n\cos\frac{n\pi x}{a}\text{ say} \quad . . . . . (2)$$

$$L_n\cosh\frac{n\pi x}{a}+\frac{x}{a}\sinh\frac{n\pi x}{a}$$
$$=-\frac{2}{n^2\pi^2}\sinh n\pi\left[1+2n^4\sum_{m=1}^{\infty}\frac{\cos m\pi}{(m^2+n^2)^2}\cos\frac{m\pi x}{a}\right]$$
$$=\beta_{on}+\sum_{m=1}^{\infty}\beta_{mn}\cos\frac{m\pi x}{a}\text{ say.} \quad . . . . . (3)$$

Then if the arbitrary edge deflexion be expanded in the interval $-a<x<a$ as $w=\gamma_o+\sum_{n=1}^{\infty}\gamma_n\cos\frac{n\pi x}{a}$, we have

$$*\gamma_n=\alpha_n+A_n(L_n\cosh n\pi+\sinh n\pi)+\sum_{m=1}^{\infty}A_m\beta_{nm}\cos m\pi,$$

whence

$$A_n=\frac{\gamma_n-\alpha_n}{L_n\cosh n\pi+\sinh n\pi}$$
$$+\frac{1}{L_n\cosh n\pi+\sinh n\pi}\sum_{m=1}^{\infty}A_m\beta_{nm}\cos m\pi. \quad (4)$$

Equation (4) may be readily solved by successive approximation to any order desired; in particular, accurate values of the first two or three coefficients, which are all that are required for very high precision, are readily obtained. Similar treatment yields the constants in the case of plates having $b\neq a$.

* Note: $\beta_{nm}=-\frac{4\cos n\pi}{\pi^2}\cdot\frac{m^2\sinh m\pi\cos m\pi}{(m^2+n^2)^2}$.

LXXXIII. *On the Determination of the Dielectric Constants of Organic Liquids at Radio Frequencies.*—Part II. *Chlorobenzene and Ethylene Dichloride. By* R. M. Davies, *M.Sc., F.Inst.P., Lecturer in Physics, University College of Wales, Aberystwyth* *.

Contents.

	Page
I. Introduction	1008
II. The Dielectric Constant Condensers	1009
III. Purification of Materials	1015
IV. Experimental Results	1017
V. Discussion of the Results and Comparison with the Results of Previous Investigations	1025
VI. List of References	1029

I. *Introduction.*

THE work described in the present paper is a continuation of that described in a previous communication [1] and has for its aim the accurate determination of the dielectric constant (d.c.) of organic liquids. For the purpose of calibrating condensers used for the determination of d.c. by comparative methods, the standard liquid employed is benzene on account of the fact that it can be thoroughly purified and that its d.c. has been determined to an accuracy of 2 in 10,000 by Hartshorn and Oliver [2]. It is, however, desirable to have what may be called secondary-standard liquids to cover the range of d.c., which extends from about 2 in the case of benzene to about 80 in the case of water. The liquids chosen in the present instance were monochlorobenzene and ethylene dichloride ; at 25° C. the d.c. of chlorobenzene is of the order of 5·6, whilst that of ethylene dichloride is of the order of 10·4. Both substances are fairly easily purified and, in addition, are chemically stable.

In the present work the method used previously has been followed and, with the exception of the d.c. condenser, the same apparatus has been used. The method used is a comparison method, and, because of the ease with which it can be purified and because of the excellent agreement between the values found by different observers for its d.c., carbon tetrachloride has been adopted as the standardizing liquid. The values found

* Communicated by Vice-Principal Gwilym Owen, M.A., D.Sc.

in the previous paper have been used ; at 20° C. the value of the dielectric constant of carbon tetrachloride is $2{\cdot}236_0$, whilst at 25° C. it is $2{\cdot}226_7$.

II. *The Dielectric Constant Condensers.*

The conditions to be fulfilled by a d.c. condenser used for comparative measurements have been discussed in Section IV. of the previous paper. The cylindrical condenser used in that work fulfilled all these conditions, with the exception, perhaps, of the condition that the solid material used for insulation should be arranged so that the electrostatic flux through the solid should be the same whatever be the d.c. of the surrounding medium. This was shown to be equivalent to the condition that there should be no refraction of the unit tubes of force passing from the insulated to the earthed plate of the condenser. The effect of any deviation from this condition would not be great in the case of the liquids (carbon tetrachloride and chloroform) used in that work, since their d.c. is of the same order as that of the standardizing liquid (benzene) ; this was shown to be the case by an additional experiment using a condenser in which the required condition is more nearly fulfilled.

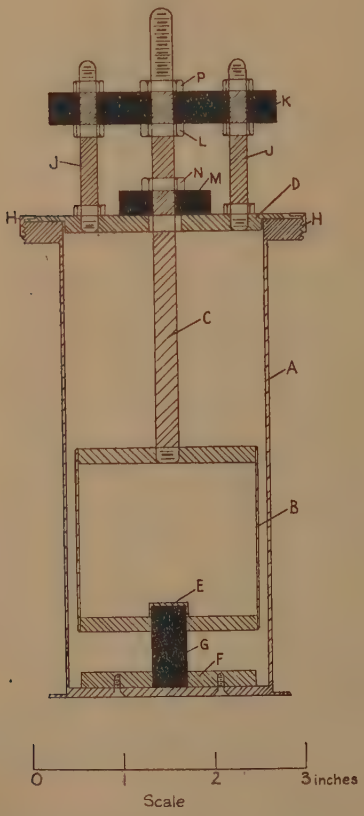
In the case of ethylene dichloride, however, failure to comply with the conditions stated may lead to considerable inaccuracy, since its d.c. is about 4·5 times as great as that of the standardizing liquid. On this account it was decided to use two condensers in which the given condition would be fulfilled as far as possible and in which the disposition of the solid dielectric would be different. One of the condensers used is a cylindrical condenser, whilst the other is a parallel-plate condenser, and the fact that they comply with the given condition is shown by the excellent agreement between the results obtained by the two condensers.

For the reasons stated in the previous paper the whole apparatus is made of brass and is chromium-plated, with the exception of the solid insulation.

The cylindrical condenser (denoted as condenser A in the sequel) is shown in section in fig. 1. The outer earthed cylinder A of 2·19 in. internal diameter, is closed at the lower end ; the inner insulated cylinder B consists of a brass tube 2 in. external diameter, and 2 in. long,

closed at its two ends by brass disks soldered in position. A brass rod C about 5 in. long, is screwed into the upper disk whilst the lower disk carries, at its centre, a short piece of brass tube E closed at its upper end. A brass

Fig. 1.



The cylindrical condenser. (Condenser A.)

disk F with a hole drilled at its centre, is fixed to the bottom of the earthed cylinder A by the two screws shown; in order to make the container liquid-tight the outer ends of these screws are covered with solder. G represents a short piece of glass rod whose two ends

are ground so as to be a tight fit in the tube E and in the hole in F.

The condenser lid D is kept in position by the ring H working on the screwed flange, which is soldered on the upper end of the earthed cylinder; a clearing hole of ample dimensions is drilled in the centre of the lid, and through this the rod C passes. Two brass pillars J are screwed into the lid and they are locked in position by the nuts shown; these pillars serve to carry an ebonite block K which is fixed firmly in position by the nuts shown. The rod C passes through a clearing hole in the centre of the block K and also through a central clearing hole in the square glass block M. The upper end of C is provided with a screw thread, and on this the three nuts L, N, and P are screwed in the positions shown.

To assemble the condenser the lid, the insulated cylinder, and the glass rod G are placed in position, the three nuts L, N, and P being slack. The nut L is now tightened up so as to press against the lower surface of K; this fixes the insulated cylinder with respect to the outer cylinder. The nut P, which serves as a lock-nut, is then tightened. Finally, the nut N is tightened, holding the glass block M tightly against the upper surface of the condenser lid.

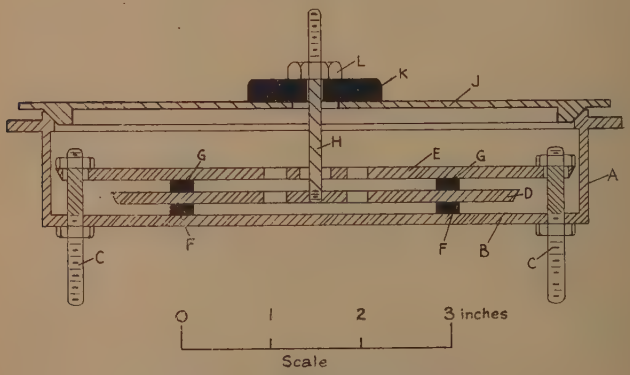
When screwed down tightly the glass block M serves to prevent excessive air-leakage from the condenser; this is essential, since a stream of pure dry air is passed through the condenser in order to keep the condenser dry before the introduction of a liquid sample and in order to prevent absorption of water by the liquid after it has been introduced.

In addition, the lid is fitted with two details which are not shown in the diagram. The first is a brass tube whose internal diameter is about $\frac{1}{4}$ in., and through this the dry-air stream is passed into the condenser. The second detail is a $\frac{1}{4}$-in. tapped hole, which is provided with a screwed cap; this is used for introducing liquid samples into the condenser.

As in the previous work the volume of liquid used in each experiment was the same, and this volume was measured by means of the special pipette described in that paper. During the course of an experiment a current of air is passed through the d.c. condenser as mentioned previously; this tends to cause evaporation of the con-

tained liquid, especially since the vessel has to remain in the thermostat for about four hours in order to complete a set of readings at 20° and 25° C. To avoid any error due to this cause it is essential to arrange matters so that a decrease of level of the liquid due to evaporation is not accompanied by any serious decrease in the capacitance of the system; in the condenser just described this can be accomplished by arranging that the volume of liquid used is such that its surface is well above the upper flat surface of the insulated cylinder, since the contribution of the liquid above this surface to the capacitance of the system is small in comparison with the contribution of the liquid below the surface. The

Fig. 2.



The parallel-plate condenser. (Condenser B.)

volume of liquid actually used was about 92 c.c., which was sufficient to fill the condenser to within an inch or so of the lid.

The total capacitance of this condenser in air was of the order of 48 $\mu\mu$ F., the replaceable air capacitance amounting to about 40 $\mu\mu$ F.

The second condenser (denoted as condenser B in the sequel) is a parallel-plate condenser, and it is shown in section in fig. 2. The outer vessel A consists of a short brass cylinder closed at its lower end by a thick brass disk B hard-soldered in position. Three brass rods C, arranged at 120° to each other, are screwed through the

lower disk B, and they are locked in position by the nuts shown ; the upper surfaces of these nuts are coated with a thin layer of low melting-point solder, and when they are screwed tightly against the lower surface of B they serve as an additional safeguard to make the vessel liquid-tight.

Three rectangular blocks, F, of fused silica, $\frac{1}{4}$ in. by $\frac{1}{4}$ in. by $\frac{1}{8}$ in., rest on B and are arranged at 120° to each other. A circular brass disk D is supported on these blocks, and into this a brass rod H is screwed. Resting on D are again three silica blocks G similar to F, and these are also arranged at 120° to each other ; in turn, these blocks support the circular brass disk E. A circular hole of ample diameter is drilled at the centre of E to allow the rod H to pass through, and, in addition, three circular holes are drilled at 120° to each other near its circumference to allow the rods C to pass through. B and E constitute the earthed plates of the condenser, whilst D is the insulated plate, the fused silica blocks F and G serving as solid insulating spacers. After this plate system has been assembled it is kept rigidly in position by means of the three nuts, which work on the upper ends of the rods C, as shown in the diagram. In order to allow free circulation of the contained liquid three holes of $\frac{1}{4}$ in. diameter are drilled at 120° to each other in the upper and middle disk, whilst the outer edges of these disks are bevelled in order to prevent the formation of air-bubbles when filling the condenser.

The condenser lid J consists of a circular brass disk to which is soldered a bevelled brass ring ; this ring beds on a corresponding bevelled edge on the upper end of the condenser vessel ; the surfaces of the ring and of the edge are ground together so as to provide a moderately air-tight joint. A circular hole of sufficient diameter is drilled in the centre of the lid, and through this hole the rod H passes. A square glass block K with a central clearing hole is slipped over the end of H, and this block can be pressed tightly on to the lid by means of the nut shown. This block, when pressing tightly on the lid, serves two functions—it provides an air-tight seal for the hole in the lid and it also keeps the lid fixed in position on the bevelled edge of the outer vessel.

As in the case of the cylindrical condenser, the lid is provided with a tube for the pure air stream and with a

hole, fitted with a screwed cap, for the introduction of liquid into the condenser. These details are not shown in the diagram.

The main portion of the capacitance of this condenser is concentrated in the plate system, and thus, by arranging that the volume of liquid used in an experiment is such that its surface is well above that of the upper condenser plate, any error due to evaporation is avoided. The volume used in an experiment was measured by the pipette arrangement and amounted to about 230 c.c.; this volume is sufficient to fill the vessel to about $\frac{1}{4}$ in. above the upper surface of the upper condenser plate.

The total capacitance of this condenser in air is of the order of 70 $\mu\mu$ F., the replaceable air capacitance amounting to about 63 $\mu\mu$ F.

It remains to discuss whether the design of the condensers is such that errors due to the refraction of unit tubes of force are negligible. Reference to the discussion in the previous work will show that there are two possible sources of error in this connexion.

In the first place, refraction of the unit tubes of force will occur in the neighbourhood of the solid insulating blocks unless these are arranged so that their surfaces are either parallel or perpendicular to the tubes of force when the condenser is filled with air. In condenser A this must be very nearly the case, since the main portion of the electrostatic flux between the lower flat surface of the insulated cylinder and the bottom of the earthed cylinder is parallel to the generators of the glass cylinder used for insulation. In addition, it is clear that the major part of the flux between the insulated and earthed cylinders is situated in the cylindrical space between them rather than in the space between their horizontal surfaces; this implies that any inaccuracy due to the effect under discussion cannot be very great. As regards condenser B the spacers are placed well within the uniform region of the electrostatic field between the earthed and insulated plates; their surfaces are either parallel or perpendicular to the unit tubes between the plates, and this source of error must therefore be absent.

In the second place, refraction of the unit tubes of force can occur near the boundaries of the electrostatic field between the earthed and insulated plate systems, and, in order to avoid errors due to this effect, it is

essential to ensure that the introduction of liquid does not alter the distribution of the unit tubes near the boundaries of the field. Owing to the fact that the insulated plate system in both condensers is completely screened by the earthed system, the only boundary that need be considered is the free horizontal surface of the liquid. The effect under discussion cannot be present at these surfaces since, in the position occupied by the liquid surface, the unit tubes are practically in a horizontal plane whether the condensers be filled with liquid or with air.

The screening and switching arrangements used in the present work were very similar to those described and used in the previous paper.

III. *Purification of Materials.*

(i.) *Carbon tetrachloride.*

This substance was purified as described in the previous paper.

(ii.) *Chlorobenzene.*

Altogether six samples were used. All six samples were given identical preliminary purification, consisting of washing once with dilute sodium hydroxide, then once with dilute sulphuric acid, and, finally, twice with water. Samples nos. 1 and 4 were then given a preliminary drying over phosphorus pentoxide, whilst samples nos. 2, 3, and 6 were given a preliminary drying over anhydrous potassium carbonate; sample no. 5 was not given a preliminary drying. It had been hoped to complete the purification process by distillation over phosphorus pentoxide or potassium carbonate in a current of dry air, using a fractionating column; it was found, however, that the distillate was always turbid, indicating that the drying agent was carried over with the distillate; this was probably due to the comparatively high boiling-point of the chlorobenzene (131·7° C.) and the length of the fractionating column (about 1 metre). For this reason samples nos. 1, 2, 5, and 6 were distilled over metallic calcium turnings, whilst samples nos. 3 and 4 were distilled without a drying agent in the distilling flask. Samples nos. 1, 2, 3, and 4 were purified from the chlorobenzene obtained from Messrs. B. D. H., and samples

nos. 5 and 6 from a supply obtained from Messrs. Hopkin & Williams.

(iii.) *Ethylene Dichloride.*

Six samples were used, and these were given the same preliminary purification as chlorobenzene. Samples nos. 1, 4, 5, and 6 were dried over phosphorus pentoxide, whilst samples nos. 2 and 3 were dried over anhydrous potassium carbonate. Samples nos. 1 and 5 were distilled over phosphorus pentoxide in a current of dry air, using a fractionating column; samples nos. 2 and 4 were similarly distilled over potassium carbonate, whilst samples nos. 3 and 6 were distilled without a drying agent in the distillation flask. Samples nos. 1, 2, and 3 were purified from the ethylene dichloride supplied by Messrs. B. D. H., and samples nos. 4, 5, and 6 from a supply obtained from Messrs. Hopkin and Williams.

As in the previous work, every possible care was taken to ensure that the samples used should be free from water, and to that end the precautions taken in that work were observed. In connexion with this aspect the solubilities of the substances used are of interest. At 30° C., according to Gross and Saylor *, a saturated solution of chlorobenzene contains 0·0488 gm. of chlorobenzene in 100 gm. water, whilst a saturated solution of ethylene dichloride contains 0·900 gm. ethylene dichloride in 100 gm. water. Assuming a linear relation between d.c. and the amount of water present, calculation shows that the d.c. of a saturated solution of chlorobenzene will be about 0·7 per cent. greater than that of the water-free substance, whilst the d.c. of a saturated solution of ethylene dichloride will be about 6 per cent. greater than that of the water-free substance. These figures serve to show that particular care must be taken to prevent access of water and moisture to the substances, especially in the case of ethylene dichloride.

As before, the refractive index of the various samples was measured in order to obtain a measure of their purity; the measurements were taken at a temperature in the neighbourhood of 20° C. and the results reduced to a

* Quoted in Landolt and Börnstein, *Physikalish-Chemische Tabellen*, Erg. iii. p. 667 (1935).

common basis of 20° C. by using the temperature coefficient obtained from Landolt and Börnstein.

IV. *Experimental Results.*

The d.c. measurements were carried out at two temperatures, one in the neighbourhood of 20° C. and the other in the neighbourhood of 25° C. Owing to the fact that it was impossible to immerse the parallel-plate condenser so that the contained liquid would be wholly below the surface of the water in the thermostat, the temperature of the liquid contained in this condenser was slightly lower than the temperature of the liquid in the cylindrical condenser, and hence the temperature of the actual liquid in each condenser was measured at the two temperatures used. At the lower temperature the temperature of the liquid in the cylindrical condenser was $19{\cdot}88 \pm 0{\cdot}03°$ C., whilst that of the liquid in the parallel-plate condenser was $19{\cdot}70 \pm 0{\cdot}03°$ C.; at the upper temperature the liquid in the cylindrical condenser was at a temperature of $24{\cdot}65 \pm 0{\cdot}03°$ C., whilst that in the parallel-plate condenser was at a temperature of $24{\cdot}41 \pm 0{\cdot}03°$ C. These temperatures were measured by a thermometer calibrated at the N. P. L.

The quantities given directly by experiment are $C'_{ra}(\epsilon_s - 1)$ for the calibrating liquid and $C'_{ra}(\epsilon - 1)$ for the other liquids, C'_{ra} being the "replaceable air-capacitance" of the condenser as defined in the previous work, ϵ_s being the d.c. of the standardizing liquid, and ϵ being the d.c. of the liquid under test. All capacitances are expressed in terms of the mean capacitance per degree rotation of the smaller variable air condenser as unit; this corresponds approximately to $1{\cdot}45\,\mu$ F.

In the case of the calibration using carbon tetrachloride the measurements were carried out at wave-lengths ranging from 860 m. to 770 m. (frequency range 350–390 Kc.). In the case of the measurements with chlorobenzene the wave-length range was 900 m. to 770 m. (frequency range 333–390 Kc.), whilst in the case of ethylene dichloride the wave-length used varied from 900 m. to 880 m. (frequency range 333–340 Kc.).

(i.) *Calibration.*

During the course of the work the value of C'_{ra} was found on six occasions, using different samples of pure

carbon tetrachloride and assuming the values quoted in Section I. for the d.c. of this substance. After the third calibration, it was found necessary to effect a small alteration in the cylindrical condenser, and for this reason the only results quoted are those obtained with the last three samples. The results obtained are summarized in Tables I. (*a*) and I. (*b*), the former relating to the cylindrical condenser and the latter to the parallel-plate condenser. The second column in the tables gives n_{20}, the refractive index (R.I.) for sodium light at 20° C., the reduction to this temperature being carried out using the temperature coefficient (-50×10^{-5}) given in the I. C. T.

TABLE I. (*a*).

Cylindrical Condenser.

Sample no.	n_{20}.	Temp.	$C'_{ra}(\epsilon_s-1)$.		C'_{ra}.	
4.....	1·46043	i	34·030		27·528	
		ii		33·724		27·476
5.....	1·46043	i	33·997		27·501	
		ii		33·728		27·479
6.....	1·46042	i	34·014		27·515	
		ii		33·740		27·489

Note :—Temp. i.=19·88° C.
Temp. ii.=24·65° C.

The mean values of $C'_{ra}(\epsilon-1)$ and of C'_{ra}, together with their probable errors calculated according to the method of least squares, are :

(i.) At 19·88° C., $C'_{ra}(\epsilon_s-1)=34{\cdot}01_1\pm0{\cdot}006$.
$C'_{ra}=27{\cdot}51_5\pm0{\cdot}005$.

(ii.) At 24·65° C., $C'_{ra}(\epsilon_s-1)=33{\cdot}73_1\pm0{\cdot}003$.
$C'_{ra}=27{\cdot}48_1\pm0{\cdot}002$.

The agreement between the individual values obtained for C'_{ra} is thus seen to be excellent.

TABLE I. (*b*).

Parallel-plate Condenser.

Sample no.	n_{20}.	Temp.	$C'_{ra}(\epsilon_s-1)$.		C'_{ra}.	
1.....	1·46059	i	53·761		43·475	
		ii		53·357		43·457
2.....	1·46046	i	53·751		43·467	
		ii		53·391		43·485
3.....	1·46037	i	53·743		43·460	
		ii		53·329		43·434
4.....	1·46043	i	53·770		43·482	
		ii		53·381		43·477
5.....	1·46043	i	53·762		43·476	
		ii		53·335		43·439
6.....	1·46042	i	53·779		43·489	
		ii		53·341		43·444

Note.—Temp. i=19·70° C.
Temp. ii=24·41° C.

The mean values and probable errors are :

(i.) At 19·70° C., $C'_{ra}(\epsilon_s-1)=53·761\pm0·004$.

$C'_{ra}=43·475\pm0·003$.

(ii.) At 24·41° C., $C(\epsilon_s-1)=53·356\pm0·006$.

$C'_{ra}=43·456\pm0·005$.

The mean value of n_{20} is $1·4604_5$, in excellent agreement with the value of 1·460 given in the I. C. T., and the value of $1·4607_2$ given in Landolt and Börnstein.

(ii) *Chlorobenzene.*

The results obtained with the six samples purified, as described in the preceding Section, are given in Tables II.(*a*) and II. (*b*), the former relating to the cylindrical condenser and the latter to the parallel-plate condenser. For the cylindrical condenser results are quoted only for the last three samples, since the condenser had to be altered between the third and fourth reading and since the value of C'_{ra} given in the previous sub-section applies only to

the condenser after this alteration. In connexion with the R.I. of this substance it is somewhat difficult to correct the observed readings accurately for temperature, on account of discrepancies in the values quoted in the tables. According to Landolt and Börnstein, vol. i. and Erg. iii., the following values are given for the R.I. for sodium light :—

(*a*) At 20° C., 1·52479. Observer : —Bruhl (1880).

(*b*) At 15° C., 1·52748. Observers :—Timmermans & Martin (1927).

(*c*) At 25° C., 1·51697. Observers :—Martin & Collie (1932).

The last two observations give $n_{20^\circ} = 1{\cdot}52223$ and the temperature coefficient as $10{\cdot}51 \times 10^{-3}$, and since one worker (Martin) has collaborated in the two results these values have been adopted.

TABLE II. (*a*).

Cylindrical Condenser.

Sample no.	n_{20°.	Temp.	$C_{vac}(\epsilon-1)$.		ϵ.	
4.....	1·52460	i	129·681		5·7130	
		ii		127·329		5·6334
5.....	1·52464	i	129·697		5·7137	
		ii		127·234		5·6298
6.....	1·52462	i	129·690		5·7135	
		ii		127·284		5·6316

Note :—Temp. i = 19·88° C.
Temp. ii = 24·65° C.

Taking mean values and probable errors it is found that :

(i.) At 19·88° C., $\epsilon = 5{\cdot}7134 \pm 0{\cdot}0001$.

(ii.) At 24·65° C., $\epsilon = 5{\cdot}6312 \pm 0{\cdot}0008$.

From these values the temperature coefficient of increase of dielectric constant $(d\epsilon/d\theta)$ is $-0{\cdot}0172_4$, giving finally

(*a*) At 20·0° C., $\epsilon_{20} = 5{\cdot}7113$.

(*b*) At 25·0° C., $\epsilon_{25} = 5{\cdot}6252$.

TABLE II. (*b*).

Parallel-plate Condenser.

Sample no.	n_{20}.	Temp.	$C'_{ru}(\epsilon-1)$.		ϵ.	
1.....	1·52454	i	205·276		5·7218	
		ii		201·505		5·6369
2.....	1·52452	i	205·112		5·7179	
		ii		201·451		5·6357
3.....	1·52469	i	205·305		5·7224	
		ii		201·610		5·6394
4.....	1·52460	i	205·218		5·7204	
		ii		201·518		5·6373
5.....	1·52464	i	205·205		5·7200	
		ii		201·479		5·6364
6.....	1·52462	i	205·345		5·7232	
		ii		201·659		5·6406

Note :—Temp. i = 19·70° C.
Temp. ii = 24·41° C.

Taking mean values and probable errors it is found that :

(i.) At 19·70° C., $\epsilon = 5{\cdot}7210 \pm 0{\cdot}0005$.

(ii.) At 24·41° C., $\epsilon = 5{\cdot}6377 \pm 0{\cdot}0005$.

From these results the value of the temperature coefficient is

$$\frac{d\epsilon}{d\theta} = -0{\cdot}0176_8,$$

whence

(*a*) At 20·0° C., $\epsilon_{20} = 5{\cdot}7157$.

(*b*) At 25·0° C., $\epsilon_{25} = 5{\cdot}6273$.

The agreement between the values found with the cylindrical condenser and the parallel-plate condenser are seen to agree to better than 1 in 1000, and taking the mean, the final values are :

(*a*) At 20·0° C., $\epsilon_{20} = 5{\cdot}713_5$.

(*b*) At 25·0° C., $\epsilon_{25} = 5{\cdot}626_3$.

(*c*) $$\frac{d\epsilon}{d\theta} = -0{\cdot}017_4.$$

The mean value of the R.I. is

$$n_{20}=1{\cdot}52460,$$

which is in good agreement with the value (*a*) quoted above, but which does not agree so well with the value interpolated from the values (*b*) and (*c*).

(iii.) *Ethylene Dichloride.*

The results obtained with the six samples purified as described above are given in Tables III. (*a*) and II. (*b*): the former relates to the cylindrical condenser and the latter to the parallel-plate condenser.

TABLE III. (*a*).

Cylindrical Condenser.

Sample no.	n_{20}.	Temp.	$C_{ra}\epsilon(\epsilon-1)$.		ϵ.	
1.....	1·44492	i	265·448		10·6474	
		ii		257·624		10·3745
2.....	1·44490	i	266·536		10·6868	
		ii		258·319		10·4001
3.....	1·44492	i	266·146		10·6730	
		ii		258·373		10·4018
4.....	1·44480	i	266·162		10·6731	
		ii		258·046		10·3901
5.....	1·44519	i	265·228		10·6394	
		ii		257·453		10·3683
6.....	1·44504	i	265·457		10·6479	
		ii		257·438		10·3681

Note :—Temp. i = 19·88° C.
Temp. ii = 24·65° C.

In the I. C. T. and in Landolt and Börnstein the following values are given for the R.I. of this substance for sodium light :—

(*a*) At 20° C., 1·44432. Observer :—Bruhl (1880).

(*b*) At 20° C., 1·44476. Observers :—Smyth, Dornte & Wilson (1931).

(*c*) At 15° C., 1·44759. Observers :—Timmermans & Martin (1928).

The mean value of (*a*) and (*b*) is 1·44454, and, combining this with (*c*), the temperature coefficient of the R.I. is found to be −0·00061 ; this value has been adopted in reducing the experimentally-determined values of the R.I. to 20° C.

Inspection of Tables III. (*a*) and III. (*b*) shows that the values obtained for the d.c. of ethylene dichloride with the two condensers and at the two temperatures

TABLE III. (*b*).

Parallel-plate Condensers.

Sample no.	n_{20}.	Temp.	$C'_{ra}(\epsilon-1)$.		ϵ.	
1.....	1·44492	i	420·650		10·6757	
		ii		408·700		10·4048
2.....	1·44490	i	421·958		10·7058	
		ii		409·101		10·4141
3.....	1·44492	i	420·871		10·6808	
		ii		408·533		10·4011
4.....	1·44480	i	421·730		10·7004	
		ii		408·940		10·4104
5.....	1·44519	i	420·264		10·6667	
		ii		408·159		10·3925
6.....	1·44504	i	420·714		10·6770	
		ii		408·466		10·3996

Note :—Temp. i=19·70° C.
Temp. ii=24·41° C.

fall into two classes ; the values obtained with samples nos. 1, 5, and 6 agree to within 1 in 1000, and the values given by samples nos. 2, 3, and 4 show similar agreement. The average value given by samples nos. 2, 3, and 4, however, exceeds that given by samples nos. 1, 5, and 6 by about 3 in 1000 ; this discrepancy is greater than the experimental error and indicates the presence of some form of systematic error. In connexion with this discrepancy it will be noted that samples nos. 2 and 4 have been distilled from potassium carbonate, whilst sample no. 3 has been dried over potassium carbonate and distilled

without a drying agent. Samples nos. 1, 5, and 6, on the other hand, have not been treated with potassium carbonate. It therefore appears that the discrepancy has to do with the treatment of the ethylene dichloride with potassium carbonate. Now, ethylene compounds are transformed into glycol by treatment with potassium carbonate solutions, and the discrepancy is therefore probably due to the presence of small amounts of glycol in samples nos. 2, 3, and 4. If this is so, then one would expect the d.c. of these samples to be greater than that of the other samples, since the d.c. of glycol is of the order of 40; this is seen to be the case.

For this reason the results given by samples nos. 2, 3, and 4 are rejected in arriving at the final value of the d.c. of ethylene dichloride.

Using the values given by samples nos. 1, 5, and 6 with the cylindrical condenser, we obtain from Table III. (*a*):

(i.) At 19·88° C., $\epsilon = 10{\cdot}645 \pm 0{\cdot}002$.

(ii.) At 24·65° C., $\epsilon = 10{\cdot}370 \pm 0{\cdot}002$.

(iii.) $$\frac{d\epsilon}{d\theta} = -0{\cdot}05765,$$

whence

(*a*) At 20·0° C., $\epsilon_{20} = 10{\cdot}638$.

(*b*) At 25·0° C., $\epsilon_{25} = 10{\cdot}350$.

Similarly, using the values given in Table III. (*b*) for the parallel-plate condenser, we obtain the following mean values and probable errors:—

(i.) At 19·70° C., $\epsilon = 10{\cdot}673 \pm 0{\cdot}002$.

(ii.) At 24·41° C., $\epsilon = 10{\cdot}399 \pm 0{\cdot}002$.

(iii.) $$\frac{d\epsilon}{d\theta} = -0{\cdot}05829;$$

giving

(*a*) At 20·0° C., $\epsilon_{20} = 10{\cdot}656$.

(*b*) At 25·0° C., $\epsilon_{25} = 10{\cdot}365$.

The results given by the two condensers are seen to agree to within 1·5 in 1000. Taking the mean of the

values given by the two condensers the final results for ethylene dichloride are :

(*a*) At 20·0° C., $\epsilon_{20}=10{\cdot}64_7$.

(*b*) At 25·0° C., $\epsilon_{25}=10{\cdot}35_8$.

(*c*) $(d\epsilon/d\theta) = -0{\cdot}057_9$.

The mean value of the R.I. (n_{20}) is $1{\cdot}4450_5$, in good agreement with the values quoted above.

V. Discussion of the Results and Comparison with the Results of Previous Investigations.

It will be convenient to consider, in the first instance, the agreement of the results for a particular condenser amongst themselves. Taking the calibration experiments, it is seen that the greatest discrepancy between the values of C'_{ra} for each condenser is about 1 in 1000. Considering the individual values obtained for the d.c. of chlorobenzene, the difference between the largest and smallest values obtained at either temperature amounts to less than 1 in 1000, and, finally, this difference is of the same order for the values which have been taken for ethylene dichloride.

Considering next the agreement between the mean values of ϵ_{20} and ϵ_{25} given by the two condensers, the agreement is within 1 in 1000 for chlorobenzene and about 1·5 in 1000 for ethylene dichloride. From this fact we may deduce that there is no serious systematic error in the measurements and, in particular, that the two condensers fulfil the conditions stated in Section II.

The results of previous determinations of the d.c. of chlorobenzene are summarized in Table IV., whilst the results for ethylene dichloride are given in Table V. The third and fourth columns of these Tables give the temperature at which the d.c's. were determined and the values obtained at these temperatures ; the fifth column gives the temperature coefficient $(d\epsilon/d\theta)$ in those instances where this quantity has been determined. The sixth column gives ϵ_{25}, the d.c. at 25° C., the reduction to this temperature being carried out using the coefficients determined in the present work in those cases where no coefficient is given. The seventh column states whether the method used was absolute (A) or comparative (C).

TABLE IV.—Chlorobenzene.

Author.	Date.	Temp. °C.	D.C.	$-\frac{d\epsilon}{d\delta} \times 10^2$.	$\epsilon_{25°}$.	A. or C.	Method.	Freq.	D.C. of benzene.
I. C. T.	1925	20	5·94	$2{\cdot}4_4$	5·82	..	..	Audio.	2·283
Walden, U., & Werner [3]	1925	13	5·65	..	5·44	C.	Res.	$6{\cdot}7 \times 10^7$	2·26
Sayce & Briscoe [4]	1926	25	5·44	..	5·44	A.	Res.	10^6	2·23
Williams & Krchma [5]	1926	25	5·610	..	5·610	A.	Res.	10^6	2·282
Kerr [6]	1926	15	$5{\cdot}68_5$	..	5·511	..	..	3×10^6	2·28
Smyth, Morgan, & Boyce [7]	1928	25	5·628	1·88	5·628	C.	Br.	5×10^5	2·273
Rolinsky [8]	1928	18 ± 1	5·723	..	5·590	C.	Res.	..	2·28
Linton & Maas [9]	1930	25	5·61	..	5·61	A.	Het.	$1{\cdot}5 \times 10^6$	2·28
Darnforth [10]	1931	30	5·41	..	5·32	A.	Br. Het.	Audio $2{\cdot}5 \times 10^5$	..
Ulich & Nespital [11]	1932	20	5·685	1·60	5·605	C.	Res.	4×10^6	2·282
Sugden [12]	1933	25	5·612	..	5·612	A.	Res.	2×10^5	2·272
Author	1935	20 25	5·714 5·626	$1{\cdot}74_4$	5·626	C.	Res.	$3{\cdot}6 \times 10^5$	2·2825 2·2726

TABLE V.—Ethylene Dichloride.

Author.	Date.	Temp. °C.	D.C.	$-\frac{d\epsilon}{d\theta} \times 10^2$.	ϵ_{25}.	A. or C.	Method.	Freq.	D.C. of benzene.
I. C. T.	1925	20	$10{\cdot}5 \pm 0{\cdot}3$	..	$10{\cdot}2 \pm 0{\cdot}3$	..	..	$10^3 - 10^6$	2·283
Harris [14]	1925	25	10·13	..	10·13	A.	Br.	10^3	2·248
Walden, U., & Werner [3]	1925	20	9·96	..	9·67	C.	Res.	$6{\cdot}7 \times 10^7$	2·26
Linton & Maas [15]	1931	25	10·38	..	10·38	A.	Res.	$1{\cdot}5 \times 10^6$	2·271
Linton & Maas [16]	1932	25	10·46	..	10·46	C.	Res.	$1{\cdot}5 \times 10^6$	2·273
Ulich & Nespital [11]	1932	20	10·5	5·2	10·24	C.	Res.	4×10^6	2·282
Sugden [12]	1933	25	10·36	..	10·36	A.	Res.	$(2 \times 10^5 - 1{\cdot}7 \times 10^6)$	2·272
Fairbrother [17]	1933	20	10·58	..	10·38	C.	Res.	10^6	2·2825
Author	1935	20 25	$10{\cdot}64_7$ $10{\cdot}35_8$	$5{\cdot}7_9$	$10{\cdot}35_8$	C.	Res.	$3{\cdot}4 \times 1{\cdot}0^5$	2·2825 2·2726

The eighth column indicates the method used—whether a bridge method (Br.), or a resonance method (Res.), or a heterodyne method (Het.). The ninth column gives the frequency at which the measurement was carried out, and the last column the value obtained or assumed for the d.c. of benzene at the temperature given in the second column. The figures in this last column are useful in estimating the accuracy of any particular determination, since the values given can be compared with those found by Hartshorn and Oliver [2]; these values are given in the rows giving the results of the present work.

Note.—In addition to the above, Hojendahl [13] has investigated the temperature variation of the d.c. of chlorobenzene over a wide range of temperature, using a heterodyne method at a frequency of approximately $1{\cdot}5 \times 10^6$. His method is a comparative method, and he calibrated his d.c. vessel using the values given in the I. C. T. for ether and ethylene dichloride. He finds that the d.c. of chlorobenzene at temperature $\theta°$ C. is given by

$$\epsilon = 6{\cdot}10\,(1 - 3{\cdot}20\,.\,10^{-3}\theta + 6{\cdot}6\,.\,10^{-6}\theta^2).$$

From this equation,

$$(d\epsilon/d\theta) = -1{\cdot}95\,.\,10^{-2} + 8{\cdot}05\,.\,10^{-5}\theta^2$$

and $\epsilon_{20} = 5{\cdot}71$; $\epsilon_{25} = 5{\cdot}61$.

Note.—Hojendahl [13] has investigated the temperature variation of the d.c. of ethylene dichloride by the method described in the foot-note to the preceding table. He finds that the d.c. of this substance at a temperature $\theta°$ C. is given by

$$\epsilon = 11{\cdot}95 - 7{\cdot}25\,.\,10^{-2}\theta - 5\,.\,10^{-6}\theta^2.$$

From this equation,

$(d\epsilon/d\theta) = -7{\cdot}25\,.\,10^{-2} - 15\,.\,10^{-6}\theta^2$; $\epsilon_{20} = 10{\cdot}50$; $\epsilon_{25} = 10{\cdot}14$

Inspection of Tables IV. and V. shows great divergence between the values found by different observers for the d.c. of chlorobenzene and ethylene dichloride. Arguing from the fact that water is likely to be the most likely impurity, one would expect the divergence to be greater in the case of ethylene dichloride; the tables, however, show that the divergence is greater in the case of chlorobenzene. In fact, in the case of ethylene dichloride, the results obtained by Linton and Maas (1931), Sugden, Fairbrother, and the author agree to within 1 in 500.

It is difficult to suggest any explanation for the discrepancies in the results for these substances, since their purification is not excessively difficult, and, in addition, the liquids have a high specific resistance, which makes the accurate determination of the d.c. much easier. It is highly desirable that further determinations of the d.c. of these liquids be carried out.

In conclusion, the author again wishes to express his indebtedness to Vice-Principal Gwilym Owen for his interest and encouragement, to Messrs. L. E. Sulston and D. Owen for their help in connexion with the construction of the apparatus, to Prof. T. C. James for the loan of the refractometer and for advice, and to Mr. C. R. Bury and Dr. Gwynne Davies for their help in connexion with the chemical aspects of the work.

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(17) Fairbrother, J. Chem. Soc. p. 1541 (1933).

November 1935.

LXXXIV. *Notices respecting New Books.*

A Treatise on Heat. By M. N. Saha and B. N. Srivastava. [Pp. xii+818.] (Allahabad and Calcutta: The Indian Press, 1935. Price 27*s.*)

IN 1931 was published the authors' 'Text Book of Heat,' of which the present volume, appropriately renamed, is the revised and extended second edition. The book is remarkable for its comprehensiveness and its balance. It is thoroughly up to date, but there are adequate accounts of the older

work; and the experimental side of the subject is presented as completely as the theoretical.

The first half of the book deals mainly with what is usually known as "Heat" in the narrower sense. Such topics as thermometry, calorimetry, and conduction are considered. There is an excellent chapter on heat engines, which includes a good account of turbines; this is particularly valuable, as it may help towards improving the status of heat as one of the subjects dealt with in physics courses. At the less advanced stages the study of heat often seems extraordinarily pointless, when it is divorced, as it so often is, from the study of the more vital applications.

The second part of the book deals with thermodynamics and statistical mechanics, and includes chapters on radiation, specific heat, and chemical equilibria. The more important thermodynamical relations are clearly derived and adequately discussed with reference to their applications. The treatment of thermodynamics, however, though it is sound, seems in some ways to fall below the high standard of the rest of the book. Much greater conciseness and elegance would be possible without any diminution in the clearness of exposition of particular points. Much would be gained merely by the adoption of the now internationally approved symbolism, and it is to be hoped that this will be possible in a future edition. The curious mixture of symbols, English and Greek, large and small, which have been allowed to survive in thermodynamical equations gives the subject, for many students, a quite illusory difficulty.

In the discussion of statistical mechanics a good account is given of Bose–Einstein and Fermi–Dirac statistics and a few of their applications. The account in the text is supplemented in some useful appendices.

Numerous references are given in footnotes and at the ends of chapters to original papers (as recent as 1935) and books, though, surprisingly, there is no reference to two well-known English books. The book, as a whole, may be said to be as complete and detailed as a general treatise could be. There are occasional lapses in phraseology, but the presentation generally is clear and straightforward. The authors are to be congratulated on having carried out a laborious and valuable piece of work with conspicuous success. The book is well printed and bound, and in price stands as a shining example.

Earthquakes and Mountains. By Harold Jeffreys, M.A., D.Sc., F.R.S. [Pp. x+183.] (Methuen. Price 7*s.* 6*d.* net.)

On questions relating to that part of geophysics which for convenience might be termed "geomechanics" the standard work of reference is undoubtedly Dr. Jeffreys's treatise

'The Earth.' Despite Dr. Jeffreys's remark in that book "if geophysics requires mathematics for its treatment, it is the earth that is responsible, not the geophysicist," it has been felt by those of limited mathematical equipment that it should not be necessary to follow through a lengthy analytical exposition in order to have some general idea of the problems treated. Although Dr. Jeffreys anticipated this objection by suggesting that the mathematical portions could in most cases be omitted by non-mathematical readers, their psychological effect as a deterrent still remains. It will therefore be a great satisfaction, both to the general reader and to the geophysicist (whether mathematically trained or not), that Dr. Jeffreys has seen fit to explain in simple language, technical in fact although not in form, what is known of the origin and maintenance of the earth's external form in relation to the interior.

The book is most attractively and lucidly written, and the predominant impression of the reader is the wealth of properly connected information that has been compressed into such a short space. Actually, there is far more behind many of the explanations than is obvious at a first reading, and, far from being merely a "popular" book in the derogatory sense of the term, this exposition will repay careful reading and cogitation. The typography is good, and there are some excellent illustrations; the quotations at the heads of the chapters, culled from 'Kai Lung's Golden Hours' and The Wallet of Kai Lung,' are as apt as those that appear in the other works of Dr. Jeffreys.

Geophysics has advanced rapidly in recent years, and the author of this book is responsible for a large share of this progress. This book may well serve as a supplement to 'The Earth' until such time as the newer matter becomes incorporated into a third edition. Further, as the author remarks in the Preface, the new book deals increasingly with the borderland between geology and geophysics, and it is very satisfactory to have the properties of solids and liquids, as known to physicists and engineers, linked up with the phenomena observed by geologists in the field and in the laboratory. This is, in fact, one of the comparatively few books that do in reality deserve the badly-overworked description of "satisfying a long-felt want."

Some Problems of Modern Meteorology. Published by the Royal Meteorological Society. [Pp. v+170.] (Price 3*s.* 6*d.* net.)

Meteorology suffers from two severe handicaps. Dealing with a wide range of phenomena, and therefore linking up with many branches of knowledge, it has received less concentrated

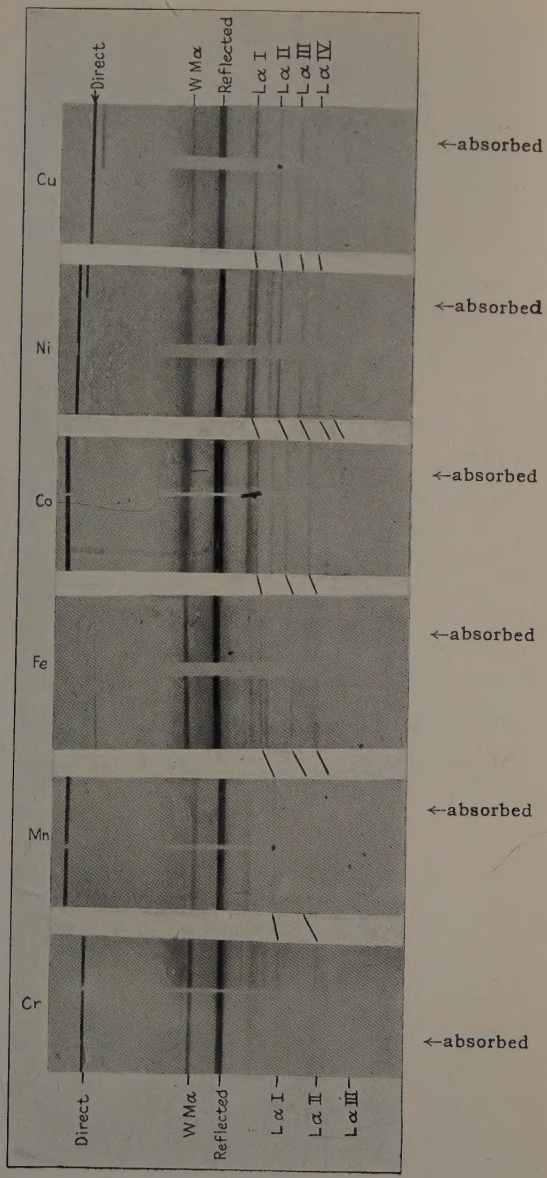
attention than some of the more compact divisions of orthodox science. Further, popular interest has centred on the prediction of comparatively small-scale phenomena, such as fine weather on a Bank Holiday, whereas with the irregular distribution of land and water on the globe some of the large-scale problems are as yet unsolved. To quote from the work under review, "We now know that the general circulation does not form an entity apart from the local circulations, however small the scale of these may be. It is not a fixed background upon which the lesser circulations are superimposed; every transitory depression or anticyclone and indeed every land or sea breeze plays its part in the determination and maintenance of the general circulation." Then, too, meteorology cannot advance, except empirically, more rapidly than the basic sciences, such as hydrodynamics, on which it depends; at the same time however, it may contribute valuable data to their progress, as in the formation of a theory of turbulence.

These considerations make a review of the present situation all the more valuable, and the position is set out in a series of sixteen papers published in the years 1930 to 1934 in the 'Quarterly Journal of the Royal Meteorological Society.' These papers, each written by a specialist under the general editorship of Prof. D. Brunt, have been reprinted and form the present volume. The papers have apparently been reproduced photographically, and the slight loss of sharpness of the print is an insignificant sacrifice to make for the advantage of very cheap publication. The controversial nature of many of the topics has led the authors to adopt a critical judicial attitude that is most helpful to those who have little time to work through the extensive literature.

To deal with particular aspects of the book would be to give an unbalanced picture of the subject; it must suffice to quote at random some of the topics that are treated. Naturally, cyclonic depressions, atmospheric circulation, the condensation of water, short-range and long-range forecasting call for critical epitomizing. The mention of meteorological acoustics, terrestrial magnetism, atmospheric ozone, and post-glacial climates indicates the wide scope of the collection. The discrimination in the summing-up is perhaps more important than the comprehensiveness; in meteorology, much more than in a self-contained subject, a theory propounded without a due sense of responsibility may prove hard to kill, and even if the corpse is decently buried, the ghost may still walk. Very possibly the value of this book in laying ghosts is as great as in directing attention to the outstanding problems of meteorology and indicating the lines of advance.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

Fig. 1.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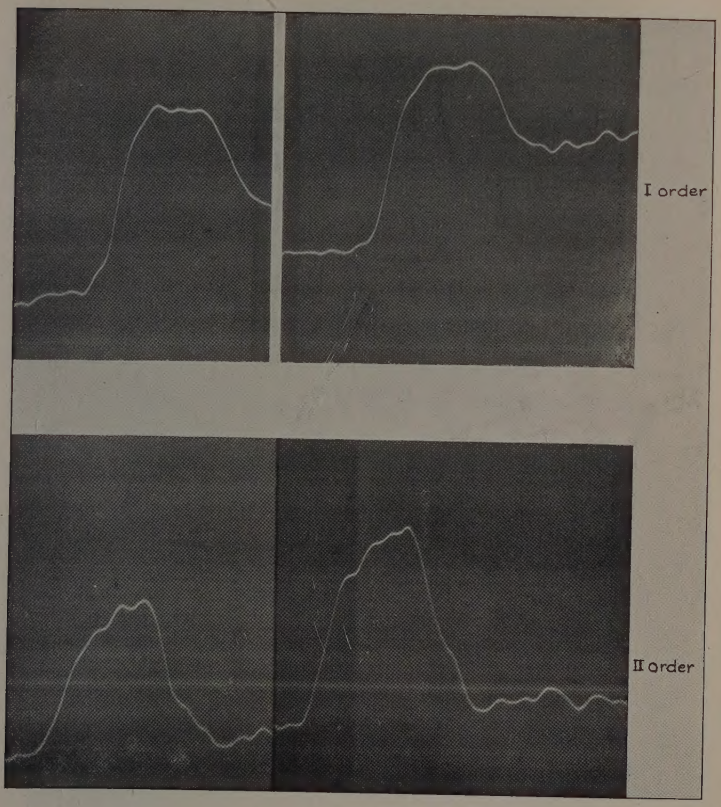
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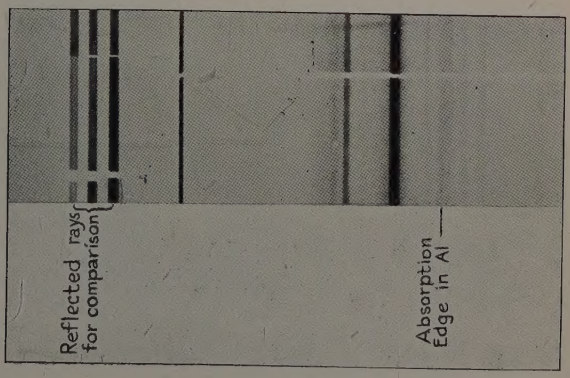
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